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Is Identity a Logical Constant and Are There Accidental Identities?[1]

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Abstract:
In the paper I discuss whether identity is an extralogical problem. Then I show that identity in Kripke's meaning when any identity implies necessary identity may be called unconditional identity and it is a apecial case of conditional. As a result, we obtain an uniform treatment of =, independent of the view whether it is a logical constants or not.

Propositional connectives and quantifiers are logical constants without any doubt. On the other hand, we speak about first-order logic with or without identity. Even this way of speaking suggests that identity has a special status to some extent. In fact, the status of identity is controversial. Wittgenstein says ([6], 5.5303):

> Roughly speaking, to say of *two* things that they are identical is nonsense, and to say of *one* thing that it is identical with itself is to say nothing at all.

According to Wittgenstein, identity is not a relation. This view rises an important point: does identity can hold between objects, which are numerically different, for example, two occurrences of 'e' in the word 'different'? Tarski's view of identity was radically opposite to that of Wittgenstein. The difference is well illustrated by the following quotation ([4], p. 49):

> Among the logical concepts not belonging to sentential calculus, the concept of IDENTITY, or, of EQUALITY, is perhaps the one which has the greatest importance.

Wittgenstein's and Tarski's statements about identity can be rephrased without essential changes by replacing 'identity as a relation' by 'identity as a predicate' (I prefer the second way of speaking).

Formally speaking, identity is introduced into first-order logic by the axioms (I omit quantifiers in the front of formulas)

(A1) $x = x$;

(A2) $x = y \Rightarrow y = x$;

(A3) $x = y \wedge y = z \Rightarrow x = z$,

together with the rule of replacement (for simplicty, I restrict it to monadic predicates, but its is obvious how to generalize this rule from arbitrary formulas)

(RR) if $(x = y)$, then $P(x) \Rightarrow P(x/y)$.

Thus, first-order logic with identity (**FOLI)** is determined by propositional calculus (**PC**, more precisely by its codification via axioms or rules of inference), pure (the meaning of 'pure' in this context will be explained below). first-order logic without identity (**PFOL**, codified by axiomatization or rules of inference) and the set {(A1)–(A3), (RR)}.

As it is well known, the identity predicate is not definable in first-order logic. The situation changes in second-order logic via the Leibniz rule:

(LR) $(x = y) \Leftrightarrow \forall P(Px \Leftrightarrow Py)$,

which says that identical objects have the same properties. In fact, the implication

(1) $(x = y) \Rightarrow \forall P(Px \Leftrightarrow Py)$

suffices for defining identity. The reverse implication

(2) $\forall P(Px \Leftrightarrow Py) \Rightarrow (x = y)$
expresses the famous *principium identitatis indiscernibilium* (the principle of identity of indiscernibles). Hence, (LR) the conjunction of (1) (the principle of indiscernability of identicals) and *principium identitatis indiscernibilium.*

Although formal properties of identity are (or seem be) clear, the concept of identity provides several problems for logicians and philosophers. One of them is captured by the already mentioned question ‘Is identity a logical constants?’. The arguments for the affirmative answer point out that fundamental metalogical results (semantic completeness, compactness, undecidability, the Löwenheim-Skolem theorem, the Lindström theorem) are valid for **FOLI**. In particular, the last results seems important because it provided a characterization of first-order logic as contrasted with higher-order logic. Consequently, the the Lindström theorem determines the borderline between ‘being the logical’ and ‘being the extralogical’, that is, the first-order thesis (see [7] for details and a discussion). Speaking metalogically, all theorems with the identity-predicate derivable in **FOLI** are universally valid. Speaking more philosophically, these theorems are necessary in the strongest sense, because logic represents an uncontroversial kind of necessity. Tarski (see [5]) argued that identity is a logical notion because it is invariant under all transformation of a domain into itself.

However, there are some problems with considering identity as a purely logical item. Having identity, we can define numerical quantifiers of the type ‘there are *n* objects’, where *n* is an arbitrary natural number. Consequently, we can characterize finite domains, although first-order logic is too weak in order to define the concept of finiteness. Now, if we add the sentence ‘there are *n* objects’ to first-order logic, its theorems are valid not universally, but in domains that have exactly *n* elements. Hence, it seems that identity brings some extralogical content to pure logic, contrary to the view (it can be expressed by a suitable metalogical theorem) that logic does not distinguish any extralogical content. Perhaps this is a very reason that the label ‘the identity-predicate’ is used, although logicians simultaneously remark that this is a very special predicate. Anyway, a qualification of identity as logical or extra-logical is conventional to some extent.

Other reason to see identity as an extralogical problem stems from so-called inflation and deflation theorems (see [3]), both closely related to the definability of finite domains in **FOLI**. The former says that if a formula, let say *A*, is satisfied in a non-empty domain **D** with *n* elements, it is also satisfied in any domain **D’** with at least *n* elements. The deflation theorem asserts that if *A* is satisfied in **D**, it is also satisfied in any **D’** with at most *n* elements. Although these theorems hold for **PFOL**, they fail for **FOLI**. The formula $\forall xy(x = y)$ provides a counterexample to the inflation theorem, because it is true in the one-element domain and no other, but the formula $\exists xy(x \neq y)$ is false in the domain with one elements. On the other hand, both theorems hold in **PFOL**. This is a reason for applying the adjective ‘pure’ to first-order logic without identity. On the other hand, if we look at **PC** and **PFOL**, we can note some metalogical diffrence betweem both systems. In particular, **PC** is decidable, bot **PFOL** has no decision procedure. Furthemore, **PC** is Post-complete, but **PFOL** (with numerical quantifiers) lacks this property. This shows that the concept denoted by the phrase ‘being the logical’ has a different strenght in particular subsystems of **PFOL**.

One additional problem requires a clarification. According to early Frege (see [1]) and Wittgenstein (see [6]), identity operates on signs. The view that the formula $x = y$ concerns objects became standard in contemporary logic. However, the notation used in (A1) – (A3) (as well as in other quoted formulas) is ambiguous to some extent. In fact, under the objectual treatment of identity, we should formulate (RR) as
(RR’) if $(d(x) = d(y))$, then $P(d(x) \Rightarrow P(x/y)$.
This formula means: if the object denoted by the letetr *x* (the denotation of *x*) is identical with the object denoted by the letter *y* (denotation of y), then if the denotation of *x* has a property *P*, the letter *x* can be replaced by the letter *y*. Note that the antecedent of (RR’) concerns object, but its consequent deals with objects and signs. A suitable rephrasing of other listed formulas is straitghtforward. The proposed reading of the replacement rule underlines its semantic character. We should thing about identity as determined by an interpretation of terms in models; denotations

depend on valuation functions ascribing objects to terms (individual constants and individual variables; I omit valuations of predicated letetrs). Note that (A1) is the only axiom, which is an unconditional formula contrary do (A2), (A3) and (RR) (or (RR'). I assume the objectual reaing of identity in what follows without using symbolism employed in (RR').

Kripke (see [2]) presnts an argument intended to show that there are no accidental identities or that every identity is necessary. This view is suported by the following reasoning. Assume (RR) and (A1) in the form (I insert quantifiers, because theis prosition plays a relevant role in the argument; the box expresses necessity):

(A1') $\forall x\Box(x = x)$ (every objects is necessarily self-identical).

Now, interprete P as the property 'necessarily identical with'. (RR) gives

(3) $\forall xy(x = y) \Rightarrow (\Box(x = x) \Rightarrow \Box(x = y)$.

Since $\Box(x = x)$ is universally valid, it can be omitted. Thus, we obtain

(4) $\forall xy(x = y) \Rightarrow \Box(x = y)$,

that is, the conclusion that if two objects are identical, they are necessarily identical. However, this result seems non-intuitive, because the identity of London and the capital of UK looks as accidental.

How convincing is this reasoning? First of all, let us change it by using the provability operator $\vdash$. Since (A1) – (A3) are logical axioms (I assume here that identity is a logical constant), we can add $\vdash$. As far as the matter concerns (RR) (translated into a formula), we obtain (quantifiers added):

(RR'') $\forall xy(\vdash((x = y) \Rightarrow (P(x) \Rightarrow P(y)))$.

Since the provability operator is monotonic, (RR'') entails

(5) $\forall xy(\vdash(x = y) \Rightarrow \vdash(P(x) \Rightarrow P(y))$.

Two things are to observed. Firstly, the andecent inside (5) has the sign $\vdash$. Is it possible to skip this element in (5). It would be at odds with the pracice of using identity in inferences. For instance, mathematicians derive conclusions about properties of identical objects, assuming that its identity is provable in mathematical theories. Secondly, we cannot interpret P as expressing provability. Now, if provability is understood as a kind of necessity, Kripke's argument cannot by repeated. We can only obtain:

(6) $\forall xy(\Box(x = y) \Rightarrow \Box(P(x) \Rightarrow P(y))$.

Inserting or omitting the formula $\Box(x = x)$ is completely pointless in this reasoning. Let us strenghten (5) to the formula

(7) $\forall xy(\vdash(x = y) \Leftrightarrow \vdash(P(x) \Leftrightarrow P(y))$.

We can think about (7) as a scheme capturing the first-order verion of the Lebniz rule. Disregarding whether the provality of the right part of (7) is realistic or not, this equivalence leads to

(8) $\forall xy(\Box(x = y) \Rightarrow \Box(x = y))$,

which is not very exciting, because it asserts that if an identity is necessary, it is necessary.

I will not discuss Kripke's solution of the puzzle produced by (5) in details (he accepts that some identities are accidental and a posteriori). I only note that his view assumes few things, in particular, the ditinnction between rigid and non-rigid designators and essentialism as well as admissibility of switching from *de dicto* necessities to *de re* ones. There is not essential difference between (A1) and (A1'), although the latter has the *de re* form (the quantifiers precede the box). However, replacing P by 'necssarily identical with' relevantly uses the *de re* formulation. My proposal conciously ignores all exrtalogical circumstances except the claim that necessity should be used *de dicto* and as related to the provability. This blocks the passing from $\Box(x = y)$ to $x = y$. Without assuming $x = y$ as independent of $\Box$ or $\vdash$, the conclusion that all identities are necessary does not follow. On the other hand, we can still keep the difference between unconditional (like (A1)) identity validities and conditional ones ((A2), (A3). Fact of interpretaion are of course accidental and a posteriori, for instance, the term 'the capital of UK' could be valued not by London, but other British city, let say, Manchester. However, if an interpretation is fixed, its consequences are conditionally necessary. Since unconditional identity is a apecial case of conditional, we obtain an uniform treatment of =, independent of the view whether it is a logical constants or not. This conclusions are obvious, if we adopt the objectual understanding of identity.

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Note:

1. This paper uses and extends some considerations in [8] and [9].