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Assertions and Conditionals: A Historical and Pragmatic Stance

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Abstract: The assertion candidate expresses a potential logical-linguistic object that can be asserted. It differs from both the act and the product of assertion; it needs not to be actually asserted and differs from the assertion made. We investigate the medieval origins of this notion, which are almost neglected in contemporary logic. Our historical analysis suggests an interpretation of the assertion candidate within the system of logic for pragmatics.
Keywords: assertion candidate, assertion sign, judgement, illocutionary force, logic for pragmatics.

1. Introduction

The theoretical nature of the notions of judgement and assertion is a key philosophical issue. A judgement is usually assumed to be the internal counterpart of an act of assertion [13] and this explains how the linguistic act of asserting is connected to the epistemic notion of judgement. In the present paper, an analysis of the concepts of judgement and assertion is provided by considering some aspects of both medieval and modern logic. Notably, we will focus on the formal system of illocutionary logic named *Logic for Pragmatics* (LP) [11], in which an assertion is justified by the existence of an *intuitive*

but *correct* proof providing conclusive evidence for the truth of a proposition. Namely, proofs in LP are considered to be *factive*.

The notion of "assertion candidate", i.e. the specific linguistic entity that *can* be asserted, has been ruled out in almost all contemporary logical systems, even if it has a long tradition in the history of logic. One of the few contemporary examples of assertion (judgement) candidate in contemporary logic is offered by the intuitionistic type theory (see [23], as observed in [40]). Our objective is to provide an interpretation for the notion of assertion candidate in LP, without introducing a specific term for it in the language of LP.

According to van der Schaar [40], the assertion candidate:

1. is different from both act and product of assertion;
2. needs not be actually asserted; it is what *can* be asserted;
3. differs from the assertion made in that it has no [assertive] force;
4. when it is expressed by a sentence *S*, it has to be explained in terms of the condition under which one is entitled to assert that *S*.

Our interpretation aims at handling the notion of assertion candidate from a pragmatic perspective in accordance with the aforementioned conditions. Moreover, a critical analysis of the medieval developments of the notion of assertion candidate will be provided.

Section 1 traces the history of the notion of assertion candidate (generally called *enuntiabile*) and some developments on language from a medieval perspective. Section 2 presents a thorough analysis of the logical system LP. Section 3 points out Russell's "embedding problem" concerning the nature of assertability and inference along with a pragmatic treatment of such problem. Section 4 outlines how to properly interpret the notion of assertion candidate in LP. Finally, Section 5 shows that certain kinds of propositions, e.g. the (empirical) undecidable ones, cannot be assumed as assertion candidates in LP. It will be pointed out that the assertion candidate is interpreted in LP as a *description* of the illocutionary act of conjecture, which is associated with certain conditions of assertability in an inferential framework.

2. Medieval Background

This section provides an outline of the medieval development of the notion of proposition. It is essential for our discussion to specify the way in which the medieval and modern philosophers understand the term "proposition" (*propositio*), in order to avoid any confusion. Ever since Russell, this term has taken the form of a *that-clause*. According to the technical use of this term in philosophical language, a proposition is not a sentence, but its cognitive content (namely, what it expressed), e.g. "Socrates amat" and "Socrates loves" express the same thought for two different grammatical sentences. In the pre-Fregean tradition, as pointed out by van der Schaar [41], propositions had the form of declarative sentences. However, in the present paper, every time we make reference to a particular historical period, we will clarify the sense in which the term proposition and other related notions are conveyed by the authors.

For medieval grammarians and logicians the term "proposition" (*propositio* or *enuntiatio*) is synonymous with speech, which can be written, spoken and mental [25]. It is not a simple issue to unravel the relation between conventional meaning of written and spoken propositions and their mental counterparts. Such an intricacy stems from the complexity of Aristotle's sentences in *De Interpretatione* (16*a* 3-8) as well as from the unanimous (but misleading) interpretation of medieval commentators.[1] The common interpretation was that a written proposition (or its mental image) conventionally signified the corresponding spoken proposition, and in turn, a spoken proposition (or its

mental image) signified the corresponding mental proposition. Thus, the written and spoken propositions accomplish their signifying function in subordination to the mental acts of apprehending and judging, always occurring before them.

An interesting position in this respect is that of Abelard [1], who emphasizes at the beginning of his commentary on Boethius' *De Interpretatione* [5] that the main object of investigation is the *propositio*, understood as a complex but unitary entity by which the name and the verb are to be conceived as components.[2] This aspect is closely connected to the distinction made by Abelard between the force and the content of an expression, which has been supported in modern logic by Frege, Russell, Geach, Searle and Vanderveken. According to Abelard, the same propositional content can be expressed with different force in different contexts: the statement that Socrates runs can be expressed by an assertion such as "Socrates runs", by a question as "Does Socrates run?", by an exclamation as "Socrates runs!", and so forth. Notably, Abelard distinguishes the assertive force of a sentence from its propositional content, a distinction that allows him to emphasize how the single components of a conditional statement are not asserted, but only the whole conditional is asserted. The statement "If Socrates is in a cell, then Socrates is a prisoner" does not assert that Socrates is in a cell and does not assert that Socrates is a prisoner, for neither the antecedent nor the consequent of an assertion of a whole conditional are asserted in this case. A pragmatic analysis of the "embedding problem" regarding the propositional contents involved in the assertion of a conditional will be provided in Section 3.

In the *Summa Logicae* (I, 1) [28] Ockham takes up the tripartite distinction of proposition,[3] pointing out that written and spoken propositions have the same meaning as the mental ones: the first two are conventional signs, while the latter is composed of natural signs. Nonetheless, he also partially restored the Aristotelian hierarchy of types of propositions since he considered the written proposition as a secondary conventional sign compared to the spoken one, and the spoken proposition as secondary and subordinate to the mental one. As a matter of fact, separate treatments of the semantics of terms and the semantics of propositions are justified by the Aristotelian distinction between two levels of speech and thought (*Categories* 1*a* 16,2*a* 4*; De Interpretatione* 16*a* 10). This means that there is (i) the level of names and verbs along with the thoughts corresponding to them, which do not yet involve any combination, and to which, therefore, it is not possible to apply the notions of truth and falsity; and (ii) the level of expressions and thoughts formed by a kind of combination that has to cope with truth and falsity (i.e. *complexio*). A *propositio* is then defined as a combination of words used to make known what is either true or false (*oratio verum falsumve significans*)[21].

It is worth noting that from the twelfth century onwards the notion of *enuntiabile* or *dictum* is occasionally used for indicating that something is assertable. Generally, the notion of *dictum* in the medieval tradition is the statement that comes in the form of a dependent clause (accusative *plus* infinitive). It is only starting from certain twelfth-century treaties that it begins to be also called *significatum*, while the expression "*enuntiabile*" appears to be an attempt to translate into Latin the expression λεχτόν of the Stoics. In the treatise *Ars Burana*, the terms *dictum*, *enuntiabile* and *significatum* are, in fact, used as synonymic terms: an *enuntiabile* is what can be asserted. In the treatise *Ars Meliduna*, assertable entities – called *enuntiabilia* – are not defined as substances or qualities, but seem to be characterized by a particular mode of being: they are inaccessible to the senses and can be merely grasped by thought.

Many medieval scholars were aware of the distinction between a *complexio* in the sense of mere predication – that is, without any assertive (or different) linguistic force – and a *complexio*, which is, instead, accompanied by an act of judging or asserting something. For instance, in the Prologue to *Ordinatio* (q. I, 55) [27], Ockham tries to clarify this doctrine and examines how the intellect can obtain knowledge through propositions.[4] Ockham distinguishes, in fact, three conditions leading to a judgement:

1. the *incomplex* cognition of the terms of a proposition (e.g., "Plato" and "man");
2. the complex cognition of terms as a whole proposition (e.g., "Plato is a man");
3. the act of assent or dissent towards the proposition, judged as either true or false (e.g., "Plato is a man is true").

Ockham also pointed out that *incomplex* cognition (of the terms of a proposition) and complex cognition (a simple act that associates a predicative-term to a subject) jointly constitute *apprehensions*. A mental proposition without an act of assent or dissent is called *notitia apprehensiva*, whereas a mental proposition containing an act of judgement (of assent or dissent) is considered a 'description' of the world and is called *notitia adhesiva* or *iudicativa*.[5] There are two types of apprehension: while the first type involves the formation of mental propositions through an act aimed at apprehending things in a compound or divided way, the second involves whole mental propositions through an act of conceiving. The act of assent or dissent is directed towards the *incomplex* (terms) or the *complex* (whole proposition), namely towards what can be true or false. From this point of view, it follows that every act of assent or dissent presupposes apprehension according to the complex as well as to the *incomplex*. Referring to the aforementioned condition 3, Nuchelmans [26] clarifies that Ockham distinguishes between two kinds of assent or dissent:

1. The first one "is the act of acknowledgement that something is the case without any reflective apprehension of the mental proposition" [26, p. 79]), leading to intuitive knowledge of something (e.g. the *acknowledgement* of the terms "Plato" and "man").
2. The second one is an act of judgement that the mental proposition is true (e.g. the *judgement* that "Plato is a man is true").

Only the second act can be properly called "judgement". This last distinction is treated in the *Quodlibetal Questions* (see III, q, 6; IV, q, 6; VII, q, 6) and also, partially, in *Reportatio* (see *Questions* 15 and 25). Ockham states that this manner of conceiving the nature of judgments is also evident from the sixth book of Aristotle's *Ethics*, where he affirmed the existence of several *habitus* like understanding, knowledge, etc.

Ockham's distinctions were generally accepted in late scholasticism. However, Jeronimo Pardo, in his *Medulla Dyalectices* [30],[6] pointed out that there must be an apprehensive cognition that needs to be different from the judicative one; in other words, a previous apprehensive cognition is required for the formation of any act of judgement.

Let us now consider Bricot's notable position [6], which has been exposed in the *Quaestiones super totam logicam Aristotelis*. In the final part, containing the commentary on the *Posterior Analytics I* of Aristotle (*Dubitatur tertio*), Bricot wonders if and how *notitia adhesiva* may be distinguished from *notitia apprehensiva*. Right after, in *Arguitur primo*, he argues that *notitia adhesiva* presupposes *notitia apprehensiva* and is posterior to it;[7] then, in *Arguitur quarto*, it is stated that the act of judgement presupposes an act of apprehension. Moreover, in the same section, it is also reported that the *notitia adhesiva*, not the *apprehensiva*, is justified by means of a proof.[8] This perspective will be relevant for our treatment of assertion, since it fulfills (at least partially) the four conditions governing the assertion candidate. Moreover, Bricot's perspective shows certain connections with our pragmatic treatment of assertions, which are, in our perspective, justified by means of proofs.

After Bricot some later authors, notably late-Scotists as well as representatives of the Thomistic school, distinguished a mere apprehensive proposition (that could be a *potential object* of *judgment* or *assertion*) from the proposition representing a judgment[9]. What has changed in the two aforementioned traditions is the name given to these types of proposition. Basically, there is no theoretical change in the two traditions but only a slight change in the terminology defining the types of propositions. A

mere apprehensive proposition is defined by the representatives of the Scotistic tradition as *propositio mentalis obiectiva*, whilst the Thomists define it as *propositio enuntiativa*. As to the type of proposition expressing a judgment, the Scotists describe it as *propositio mentalis formalis*, while the Thomists regard it as *propositio iudicativa*.

3. Logic for Pragmatics: the Elements

The distinction between judgement and predication is essential for the pragmatic analysis of sentences, which was also suggested by Frege [16].[10] According to Frege, there is a clear distinction between thoughts and judgements. Namely, a thought has a truth value, while a judgement is the acknowledgment of the truth of a thought. An assertion is the external counterpart of a judgement and is expressed by mean of a specific sign. The assertion sign "⊦" consists of two parts: the horizontal stroke "-", expressing that the content is judgeable and the vertical stroke "|", indicating that an assertion has been made. In Frege's system, no assertive sentence contains more than one pragmatic sign. Reichenbach observed that assertions are part of the pragmatic aspects of language and cannot be connected by means of truth-functional connectives.[11] He also pointed out that "assertion" is used in three different ways:

1) it denotes, first, the act of asserting
2) the result of this act, i.e., an expression of the form "⊦*p*"
3) a statement which is asserted, i.e., a statement "*p*" occurring within an expression "⊦*p*".

At any rate, there is no explicit explanation of what counts as a *potential "object" of assertion*, contrary to what occurred in medieval logic.

In LP, ⊦ *p* may be unjustified if there is no conclusive evidence confirming *p*. Therefore, *p* does not necessarily stand for an entity that *can* be part of a justified assertion when the possibility to get complete evidence (proof) is ruled out. Dalla Pozza and Garola, in their logical system named *Logic for Pragmatics* (LP) [11], provided a formal treatment of assertion, introducing pragmatic connectives. They suggested a pragmatic interpretation of intuitionistic propositional logic as a pragmatic logic for assertions, inspired by Austin's views on assertion (see [2]) and Searle and Vanderveken's theory of illocutionary logic [37]. According to the LP system, propositions can be either true or false, while the judgements expressed as assertions can be justified (*J*) or unjustified (*U*). Assertions in LP are purely logical entities, without any reference to the speaker's intentions or beliefs [11].

LP consists of two sets of formulas: radical and sentential. Interestingly, every sentential formula contains at least a radical formula as a proper sub-formula. Radical formulas are semantically interpreted by assigning them a (classical) truth value, while sentential formulas are pragmatically evaluated by assigning them a justification value (*J*, *U*) defined in terms of the intuitive notion of proof. The pragmatic language LP is the following:

Descriptive signs: propositional letters: $p, \gamma_{1}, \gamma_{2}, \ldots$
Logical signs for radical formulas: ¬, ∧, ∨, →, ↔.
Logical signs for sentential formulas: the sign of pragmatic illocutionary force ("⊦" assertion);
the pragmatic connectives: ~ pragmatic negation, ∩ pragmatic conjunction, ∪ pragmatic disjunction, ⊃ pragmatic implication, ≡ pragmatic equivalence.

Formation Rules (FRs)

Radical formulas (*rfs*) are recursively defined by the following FRs:

FR1 (atomic formulas): every propositional letter is a *rf*

FR2 (molecular formulas):

(i) Let γ be a *rf*, then $\neg \gamma$ is a *rf*
(ii) Let γ_1 and γ_2 be *rfs*, then $\neg\gamma$, $\gamma_1 \wedge \gamma_2$, $\gamma_1 \vee \gamma_2$, $\gamma_1 \rightarrow \gamma_2$, $\gamma_1 \leftrightarrow \gamma_2$ are *rfs*

Sentential formulas (*sfs*) are recursively defined by the following FRs:

FR3 (elementary formulas): Let γ be a *rf*, then $\vdash \gamma$ is a *sf*

FR4 (complex formulas):

(i) Let δ be a *sf*, then $\sim\delta$ is a *sf*
(ii) Let δ_1 and δ_2 be *sfs*, then $\sim \delta$, $\delta_1 \cap \delta_2$, $\delta_1 \cup \delta_2$, $\delta_1 \supset \delta_2$, $\delta_1 \equiv \delta_2$ are *sfs*.

Thus, every radical formula of LP has a truth value. And every sentential formula has a justification value that is defined in terms of the intuitive notion of proof and depends on the truth value of its radical sub-formulas. The semantics of these radical formulas is the same as for classical logic: it provides the interpretation for the radical formulas, by assigning them a truth value and for propositional connectives as truth functions in the standard way. LP has both a classical fragment (CLP) and an intuitionistic fragment (ILP). CLP is the fragment of LP without pragmatic connectives.

Axioms for CLP are the following:

$$A_i \vdash (\gamma_1 \rightarrow (\gamma_2 \rightarrow \gamma_1))$$

$$A_{ii} \vdash ((\gamma_1 \rightarrow (\gamma_2 \rightarrow \gamma_3)) \rightarrow ((\gamma_1 \rightarrow \gamma_2) \rightarrow (\gamma_1 \rightarrow \gamma_3)))$$

$$A_{iii} \vdash ((\neg \gamma_2 \rightarrow \neg \gamma_1) \rightarrow ((\neg \gamma_2 \rightarrow \gamma_1) \rightarrow \gamma_2))$$

Modus ponens rule for CLP is:

[MPP] if $\vdash \gamma_1$, $\vdash (\gamma_1 \rightarrow \gamma_2)$, then $\vdash \gamma_2$

The semantic rules are the commonly used classical Tarskian rules σ; thus, regulating the semantic interpretation of LP. Let γ_1, γ_2 be radical formulas and 0 = false and 1 = truth; then:

(i) $\sigma(\neg \gamma_1)=1$ iff $\sigma(\gamma_1)=0$
(ii) $\sigma(\gamma_1 \wedge \gamma_2)=1$ iff $\sigma(\gamma_1)=1$ and $\sigma(\gamma_2)=1$
(iii) $\sigma(\gamma_1 \vee \gamma_2)=1$ iff $\sigma(\gamma_1)=1$ or $\sigma(\gamma_2)=1$
(iv) $\sigma(\gamma_1 \rightarrow \gamma_2)=1$ iff $\sigma(\gamma_1)=0$ or $\sigma(\gamma_2)=1$

Pragmatic connectives have a meaning, which is explicated by the *BHK* (Brouwer, Heyting, Kolmogorov) intended interpretation of intuitionistic logical constants. The philosophical interpretations of intuitionistic logic are, in fact, provided by means of notions like assertion, construction, problem.

The illocutionary force of assertion plays an essential role in determining the *pragmatic* component of the meaning of an elementary expression, together with the *semantic* component, i.e. the meaning expressed by radical formulas.

A pragmatic interpretation of LP is an ordered pair $<\{J, U\}, \pi_\sigma>$, where {J, U} is the set of justification values and π_σ is the function of pragmatic evaluation in accordance with the following justification rules:

Justification Rules: They regulate the pragmatic evaluation π_σ, specifying the justification-conditions for the assertive formulas in function of the σ-assignments of truth-values for their radical sub-formulas:

JR1 – Let γ be a radical formula. $\pi_\sigma(\vdash\gamma)$= J iff a proof exists that γ is true, i.e. that σ assigns to γ the value «true». $\pi_\sigma(\vdash\gamma)$= U iff no proof exists that γ is true.

JR2 – Let δ be an assertive formula. Then, $\pi_\sigma(\sim\delta)$= J iff a proof exists that δ is unjustified. i.e., that $\pi_\sigma(\delta)$= U.

JR3 - Let δ_1 and δ_2 be assertive formulas. Then:

(i) $\pi_\sigma(\delta_1 \cap \delta_2)$= J iff $\pi_\sigma(\delta_1)$=J and $\pi_\sigma(\delta_2)$=J;
(ii) $\pi_\sigma(\delta_1 \cup \delta_2)$= J iff $\pi_\sigma(\delta_1)$=J or $\pi_\sigma(\delta_2)$=J;
(iii) $\pi_\sigma(\delta_1 \supset \delta_2)$= J iff a proof exists that $\pi_\sigma(\delta_2)$=J whenever $\pi_\sigma(\delta_1)$=J.

The *Soundness Criterion* (SC) is the following:

Let be γ a *rf*, then $\pi_\sigma(\vdash\gamma)$= J implies that $\sigma(\gamma)$=1.

SC states that if an assertion is justified, then the content of assertion is true. It is evident from the justification rules that sentential formulas have an intuitionistic-like formal behaviour and can be translated into the modal system S4, where □γ means "there is an (intuitive) proof (conclusive evidence) for γ". A formula δ is pragmatically valid or p.valid (respectively invalid or p.invalid) if for every Tarskian semantic interpretation σ and for every pragmatic function of justification π_σ it follows that $\pi_\sigma(\delta)$=J (respectively $\pi_\sigma(\delta)$ =U). Notice that the intuitionistic fragment of LP, ILP, is composed by complex formulas with atomic radicals [11]. The axioms of ILP are the following:
ILP Axioms:

A1. $\delta_1 \supset (\delta_2 \supset \delta_1)$
A2. $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\delta_2 \supset \delta_3)) \supset (\delta_1 \supset \delta_3))$
A3. $\delta_1 \supset (\delta_2 \supset (\delta_1 \cap \delta_2))$
A4. $(\delta_1 \cap \delta_2) \supset \delta_1$; $(\delta_1 \cap \delta_2) \supset \delta_2$
A5. $\delta_1 \supset (\delta_1 \cup \delta_2)$; $\delta_2 \supset (\delta_1 \cup \delta_2)$
A6. $(\delta_1 \supset \delta_3) \supset ((\delta_2 \supset \delta_3) \supset ((\delta_1 \cup \delta_2) \supset \delta_3))$
A7. $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\sim\delta_2)) \supset (\sim\delta_1))$
A8. $\delta_1 \supset ((\sim\delta_1) \supset \delta_2)$

Modus ponens rule for ILP is:

[MPP'] if δ_1, $\delta_1 \supset \delta_2$, then δ_2

where δ_1 and δ_2 contain atomic radicals.

It is worth noting that justification rules do not always allow the determination of the justification value of a complex sentential formula when all the justification values of its components are known. For instance, $\pi_\sigma(\delta)$=J implies π_σ (~δ)=U but not *vice versa* and π_σ (~δ)=J implies $\pi_\sigma(\delta)$=U but not *vice versa*. In addition, a formula δ is pragmatically valid or p.valid (respectively invalid or p. invalid) if for every π and σ, the formula $\pi_\sigma(\delta=)$ J (respectively π_σ (δ) =U). Hence, no principle analogous to the truth-functionality principle for classical connectives holds for the pragmatic connectives in LP, since pragmatic connectives are governed by *partial* functions of justification.

The modal (semantic) projection ( )* of pragmatic assertions is provided by the following translation in the modal system S_4, which provides a modal description of the pragmatic and illocutionary acts of assertion[12]:

$$(\vdash \gamma)^* \qquad \Box\, \gamma$$

$$(\sim \delta)^* \qquad \Box \neg\ (\delta)^*$$

$$(\delta_1 \cap\ \delta_2)^* \qquad (\delta_1)^* \wedge (\delta_2)^*$$

$$(\delta_1 \cup \delta_2)^* \qquad (\delta_1)^* \vee (\delta_2)^*$$

$$(\delta_1 \supset \delta_2)^* \qquad \Box\, ((\delta_1)^* \rightarrow (\delta_2)^*)$$

Notice that assertions, as acts, can be justified or unjustified, while the modal formulas are a *description* of assertions which can be true or false.

Classical and intuitionistic formulas are formally related by means of the following "bridge principles" connecting classical and pragmatic connectives [11]:

(a) $(\vdash \neg \gamma) \supset\ (\sim(\vdash \gamma))$
(b) $((\vdash\gamma_1) \cap (\vdash \gamma_2)) \equiv (\vdash(\gamma_1 \wedge \gamma_2))$
(c) $((\vdash\gamma_1) \cup (\vdash \gamma_2)) \supset (\vdash(\gamma_1 \vee \gamma_2))$
(d) $(\vdash(\gamma_1 \rightarrow \gamma_2)) \supset (\vdash\gamma_1 \supset \vdash \gamma_2)$

It is worth noting that (a) – (d) show the formal relations between classical truth-functional connectives and pragmatic connectives. (a) states that the assertion of a negated proposition entails the pragmatic negation of the assertion, (b) shows that the conjunction of assertions is equivalent to the assertion of the conjuncts, (c) states that a disjunction of assertions implies the assertion of the disjuncts, while (d) indicates that truth-conditional implication implies pragmatic implication.[13]

4. Russell's "Embedding Problem": the Nature of Proof and the Inferential Role of Assertions

The notion of assertion has a fundamental inferential role in logic. For instance, a variety of perspectives on proofs and assertions has been carried out in recent years in constructivism. Proofs can be viewed in a myriad of ways; namely, as objects or processes, logical or empirical, temporal or eternalist, mind-dependent or absolute[14]. Although proofs provide a complete justification for assertive judgments, from an antirealist perspective, proofs are assumed to epistemically constrain (or to be equivalent to) intuitionistic truths. On the other hand, antirealist viewpoints do not always properly distinguish the semantic notion of truth from the pragmatic *criterion*, which is geared towards creating a method that establishes the truth value of a proposition. Unlike antirealists, Dalla Pozza & Garola [11] hold that what can be actual or potential is the pragmatic notion of judgement (assertion), not the semantic notion of truth, which relies on an eternalist perspective.

An important philosophical and logical problem seems to be associated with the conditions of assertability (and provability) in an inferential framework. Russell [38], in fact, observed that there is something odd in the standard account of *Modus Ponens*, namely in the inference: p, $p \rightarrow q$; therefore q: "the proposition "p implies q" *asserts* an implication, though it does not *assert* p or q.[15] The p and the q entering into this proposition are not strictly the same as the p or the q, which are separate propositions" (see [38, p. 35])[16]. This problem is named "embedding problem" and has a variant in meta-ethics known as the Frege-Geach problem[17] [16], [17]. As stated before, the *Modus ponens* rule in CLP is represented by the following argument:

MPP:

$(i^*) \vdash p$
$(ii^*) \vdash (p \rightarrow q)$
$\therefore (iii^*) \vdash q$

while in ILP, the *modus ponens rule* is represented by this different argument:

MPP':

$(i) \vdash p$
$(ii) \vdash p \supset \vdash q$
$\therefore (iii) \vdash q$

Note that (ii*) implies (ii) (bridge principle (d)). It is not difficult to show that Russell's objection to MP does not hold in the intuitionistic fragment of LP [12]. Notice that such objection can be overcome even in (an extension of) CLP, by making use of the bridge principle (d). In fact, at any step of an inference of LP, a logical law can be introduced. If we introduce (d) as an additional premise, then it follows that:

$(1) \vdash p$	premise
$(2) \vdash (p \rightarrow q)$	premise
$(3) (\vdash (p \rightarrow q)) \supset (\vdash p \supset \vdash q)$	bridge principle (d)
$(4) \vdash p \supset \vdash q$	from (*2*), (*3*) and MPP
$(5) \vdash q$	from (*1*) and (*4*) and MPP'

Henceforth, Russell's objection to the "embedding problem" can be overcome in LP. There exists, indeed, a kind of priority in the intuitionistic approach in the theory of deduction over the classical one in LP since inferences occur between assertions (judgments) that have an intuitionistic-like formal behaviour, and not between classical propositions. Moreover, it is also remarkable that (4) is justified by the rule JR3.3 for the intuitionistic conditional when the antecedent is also justified. Thus, unjustified antecedents in a conditional cannot justify a conditional.

5. The Assertion Candidate and Logic for Pragmatics

This section tries to provide an answer to the following question: can the illocutionary force be unrestrictedly applied to propositions in order to have justified assertions? In our framework, we will point out that assertion candidates can be interpreted in a pragmatic framework as (semantic and modal) descriptions of conjectures. A conjecture may be converted into a justified assertion if the proof (conclusive evidence) for its content becomes available. The description of a conjecture can be *potentially* asserted even if they may be *not effectively* asserted[18]. In this respect, it is important to

distinguish *intrinsically undecidable propositions* (IUPs) from *contingently undecided propositions* (CUPs)[19]. On the one hand, if propositions are intrinsically undecidable then they cannot be converted into assertions even in line of principle; therefore, they cannot work as assertion candidates since they always remain unjustified. On the other hand, contingent undecided proposition might be converted into proper assertions when new evidence is available. An example of intrinsically undecidable proposition is due to Pap [29, p. 37][20]: "everybody in the universe, including our measuring rods, is constantly expanding, the rate of expansion being exactly the same for all bodies."

Pap's sentence is clearly not verifiable, even if it has a truth condition, namely that we know how the world should be in order to make the sentence true. Hence, Pap's sentence cannot be asserted and justified in line of principle and cannot work as an assertion candidate, for instance, as the antecedent of a conditional in a *BHK* framework. On the other hand, there are contingently undecided propositions that may possibly be verified, for instance in the future, such that they can be converted into assertions or can remain forever unknown.

The possibility of assertion in our pragmatic framework shows some similarities with the illocutionary act of conjecturing. In fact, the assertive sign is translated and described as □ in the modal system S4, the act of hypothesizing with ◊, while the act of conjecture can be translated as ◊□ in S4 (see [3]).[21] The modal translation is very important for our proposal, since it may provide a semantic descriptive version of a pragmatic act, thus expressible in the radical part of a sentential formula. Genuine illocutionary operators cannot be, in fact, nested [16] but, of course, the semantic description of an illocutionary act can be part of a radical formula. The assertion candidate in this case may be, thus, interpreted as the (modal) radical part of a sentential formula as $\vdash \Diamond\Box\, p$.[22] The radical part of this formula states that there exists the possibility to prove (and, therefore, to assert) *p*.

Let us take a closer look at the pragmatic notion of conjecture. A hypothesis is justified if there exists the epistemic possibility (at least a *scintilla of evidence*) grounding the content [10], while a conjecture expresses the *possibility to assert* the content (being not intrinsically undecided). For instance, Goldbach's Conjecture can be viewed as a statement that is actually not proven, even if there is the possibility that it may be demonstrated in the future.[23] On the contrary, since it is impossible to prove an *intrinsically undecidable proposition* and its (classical) negation, then these conditions can be expressed in LP in this way:

$$\text{IUP: } \sim \vdash \gamma \cap \sim \vdash \neg \gamma$$

The first conjunct of IUP can be translated in S4 as $\Box\neg\Box\, \gamma$, that is equivalent to $\neg\, \Diamond\Box\, \gamma$ (namely, the description of the fact that the conjecture γ is unjustified), while the second conjunct of IUP is translated in S4 as $\Box\neg\Box\neg\, \gamma$, that is equivalent to $\neg\, \Diamond\Box\neg\, \gamma$ meaning that the description of the fact that the conjecture $\neg\, \gamma$ is unjustified. Therefore, if γ is an intrinsically undecidable proposition, then γ as well as $\neg\, \gamma$ are not conjecturable. On the other hand, a contingently undecided proposition might be subsequently conjecturable once new evidence justifies the possibility of the assertion of the corresponding content. A contingently undecided proposition can be expressed in the meta-language of LP as:

$$\text{CUP: } \pi_\sigma (\vdash \neg\, \gamma) = U \text{ and } \pi_\sigma (\vdash \gamma) = U$$

The modal translations of CUP are $\neg\, \Box\, \neg\, \gamma$ and $\neg\, \Box\, \gamma$, respectively $\Diamond\, \gamma$ and $\Diamond\neg\, \gamma$. This explains the contingency of such kind of propositions. Following our pragmatic interpretation, it seems that what can be asserted may be interpreted as the specific description of a conjecture, and, when applying the modal translation, also as a modal radical formula.

Indeed, our pragmatic interpretation of the assertion candidate fulfills van der Schaar's four conditions, namely:

(1) it is different from both the assertion act (⊦) and the assertion product (⊦ *p*);
(2) it is what *can* be asserted in principle;
(3) it differs from the assertion made in that it has no [assertive] force;
(4) it is expressed in terms of the conditions of possibility under which one is entitled to assert a sentence.

Condition 1 pragmatically expresses the fact that it is necessary not to collapse the assertion act into the assertion product since the latter is the assertion sign *plus* the radical formula. The description of a conjecture expresses the (modal) radical formula that may be asserted. Condition 2 expresses that, from a pragmatic perspective, intrinsically undecidable propositions cannot work as assertion candidates since their possibility to be asserted is *a priori* ruled out. Condition 3 states that the assertion candidate, intended as the *description* of a conjecture, works as a specific radical formula, which does not have an assertive force, but that may be asserted and justified in case of conclusive evidence. Finally, condition 4 indicates the conjecturable nature of the content of an assertion, working as an assertion candidate.

6. Conclusion

The notion of assertion candidate has not been thoroughly analyzed in contemporary systems of logic. To meet this need, we have provided an interpretation of this notion within the framework of LP. Our pragmatic interpretation has been suggested by the medieval treatment of the notion of *enuntiabile*. The *enuntiabile*, in fact, is different from the act of assertion: it is not the result of such act but is rather what *can* be asserted.

Differently from other logical systems, in LP it is possible to distinguish the assertive force from its content and provide a way out of Russell's "embedding problem", which in itself is a major concern in philosophy of logic due to the justification of logical inference.

Our pragmatic treatment has clarified that assertion candidates can be interpreted as specific radical formulas that express a modal proposition with a possibility of being asserted. In LP, we have argued that assertion candidates can be viewed as the *description* of conjectures. However, we do not claim that assertion candidates are identical to conjectures, but rather that they can be interpreted in LP as the (modal) description of a conjecture.

Our analysis has also elucidated that IUPs cannot be asserted even in line of principle, whereas CUPs are those unjustified propositions that when integrated with new evidence do not reveal whether they may or may not become justified assertions. Lastly, both IUPs and CUPs have received a formal treatment in LP in order to isolate their logical and pragmatic features.

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Notes

1. In particular, see the Latin translation of Boethius' *De Interpretatione* [5];[18] (cfr. Boethius i 7.10 ff. in Meiser 1880): "Sunt ergo ea quae sunt in voce earum quae sunt in anima passionum notae, et ea quae scribuntur eorum quae sunt in voce. Et quemadmodum nec litterae omnibus eaedem, sic nec eaedem voces; quorum autem hae primorum notae, eaedem omnibus passiones animae sunt, et quorum hae similitudines, res etiam eaedem."
2. [1, pp. 307 (20-23) and 207].
3. Starting from the studies on obligationes, a distinction between different attitudes, namely *affirmo*, *nego*, and *dubito*, was quite common in Middle Age.
4. Being the medieval notion of proposition distinguished into mental, written and spoken, then the term "knowledge" is used here not as a substitute for the modern sense of propositional knowledge but in a broader sense.
5. "Est igitur prima distinctio ista quod inter actus intellectus sunt duo actus quorum unus est apprehensivus, et est respectu cuiuslibet quod potest terminare actum potentiae intellectivae, sive sit complexum sive incomplexum; quia apprehendimus non tantum incomplexa sed etiam propositiones et demonstrationes et impossibilia et necessaria et universaliter omnia quae respiciuntur a potentia intellectiva. Alius actus potest dici iudicativus, quo intellectus non tantum apprehendit obiectum sed etiam illi assentit vel dissentit. Et iste actus est tantum respectu complexi, quia nulli assentimus per intellectum nisi quod verum reputamus, nec dissentimus nisi quod falsum aestimamus. Et sic patet quod respectu complexi potest esse duplex actus, scilicet actus apprehensivus et actus iudicativus." […] "Prima conclusio praeambula est ista quod actus iudicativus respectu alicuius complexi praesupponit actum apprehensivum respectu eiusdem" (Ockham, *Scriptum in libros sententiarum*, *Prologue*, I, 1, O. Q.)
6. *Medulla Dyalectices* was written in 1505.
7. "notitia adhesiva presupponit notitia apprehensiva et est ea posterior, sed nihil est posterius semel positio. Igitur notitia adhesiva distinguitur ab apprehensiva" (Bricot, *Quaestiones super totam logicam Aristotelis*, Y, 5).
8. *Ibidem*: "Quarto sic notitia apprehensiva conclusionis non acquiritur per demonstrationem cum aliquum sit ante eam et tamen eius notitia adhesiva per eam acquiritur. Igitur notitia adhesiva distinguitur a notitia apprehensive."
9. See for Scotists, [39]; for Thomists in particular, see [19].

10. We focus on the Frege's distinction between predication and judgments. However, Descartes and Spinoza also previously discussed such distinction. These notions have played an important role also for the foundations of mathematics, especially at the origins of intuitionism [22].
11. See section 57, called "Logical terms in a pragmatic capacity", in [36]. However, we do not claim that there is direct influence of Frege's perspectives on Reichenbach as it happened to be with the Vienna Circle.
12. The modal translation of the pragmatic connectives in S4 is the standard one presented in [11].
13. Extensions and applications of LP to philosophical issues have been presented in [3], [4], [7], [9], [10].
14. The main antirealist perspectives regarding the nature of proofs and assertions are those of Dummett, Prawitz and Martin-Löf. Let us consider, first, Prawitz's picture on proof: "[A] mathematical sentence is true if there is a proof of it, in a tenseless or abstract sense of exists […]. That we can prove A is not to be understood as meaning that it is within our practical reach to prove A, but only that it is possible *in principle* to prove A" [33, pp. 153-154]. Dummett's replied that: "We can introduce such a notion [of eternalist proof] only by appeal to some platonistic conception of proofs as existing independently of our knowledge, that is, as abstract objects not brought into being by our thought" [14, pp. 258-9]. A more sophisticated version of antirealism is expressed by Martin-Löf: "That a proposition A is actually true means that A has been proved, that is, that a proof of A has been constructed, […] whereas to say that A is potentially true means that […] a proof of A can be constructed" [23, p. 142].
15. For instance, notice that in LP also the assertion of a disjunction is not equivalent with the disjunction of assertions. Russell's presentation of the embedding problem is done by considering Carroll's paradox of inference. For a recent work on it, see [24].
16. Quine [34, p. 12] states that "an affirmation of the form "if *p* then *q*" is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent", given the antecedent. In any case, some counterexamples to Quine's idea are easily conceivable.
17. We have pointed out that the distinction between assertive force and propositional content was already described in Abelard's writings.
18. One might wonder if the linguistic declarative mood *correlates* with or *encodes* with the type of force, notably the assertive force. By contrast, according to Recanati "declarative sentences do not correlate with any category of illocutionary force. They are illocutionarily neutral. A declarative sentence represents a state of affairs, […]; how the representation is interpreted (in illocutionary terms) is left to context. (Of course there is a blocking effect due to the competition with the other moods — those which do correlate with types of illocutionary force)" [35, p. 630]. Quite different is Dummett's perspective for which it is important to distinguish the mere *act* of assertion from the *point* with which the assertion is performed [13].The aim of assertion is truth, and assertions need to be publically recognized as such in order to be justified; nonetheless, assertions in natural language can be performed to convey different attitudes (points) of statement use.
19. Such distinction has been partially inspired by Dummett [15].
20. See also ([34], section II.1). A similar sentence is also mentioned in [20].
21. A stronger view on conjecture is expressed in [29], where a conjecture is expressed as $\Diamond\Box$ in the modal system S4.2.
22. For an extension of LP with modal formulas as radicals, see [8].
23. Of course, Goldbach's conjecture may also remain unsolved in the future or it may happen that a new theorem might state the impossibility to prove the content of the conjecture. An example of 'impossibility' theorem is given by the Abel–Ruffini theorem, stating that there is no general algebraic solution to polynomial equations of degree ≥ 5.