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A Note on Intended and Standard Models

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Abstract:
This note discusses some problems concerning intended, standard, and non-standard models of mathematical theories. We pay attention to the role of extremal axioms in attempts at a unique characterization of the intended models. We recall also Jan Woleński's views on these issues.
Keywords: intended model, standard model, extremal axiom, metalogic, categoricity, completeness, Jan Woleński.

1. The Distinction: Intended Model versus Standard Model

Mathematical theories may concern either a specified structure or a class of structures. Examples of theories of the first kind include theories of fundamental number systems (natural numbers, integers, rational numbers, real numbers, complex numbers), certain systems of geometry (for instance Euclidean geometry), and possibly also set theory, at least at the early stage of its development. Theories of the second kind include theory of groups, fields, topological spaces, vector spaces, and so on. The distinction in question applies to modern mathematics, it does not make sense in the case of mathematics before the second half of the 19th century.

The notions of intended, standard and non-standard models may be applied in the case of theories of the first kind, for obvious reasons. The terms 'intended model' and 'standard model' are used sometimes interchangeably in literature. I propose to distinguish them in the following manner. The intended model of a theory is a structure which motivated the development of the theory in question. As a rule, this structure has been investigated for a long time and its properties are based on well-established mathematical intuitions emerging from the research practice. A necessary condition for a structure to become an intended model is thus its domestication in the mathematical research. One could also say that intended models are cognitively accessible to a high degree. Then there emerges a theory of such a structure, ultimately an axiomatic theory.

The above characterization of the concept 'intended model' is intuitive, which in turn implies that the concept itself is also intuitive. A prominent example of an intended model in this sense is the natural number series with arithmetical operations defined in the usual way. Rational, real and complex numbers (as understood before the construction of the corresponding axiomatic theories of such numbers) provide further examples. It seems that the universe of the naive set theory could also be considered an example in this respect.

The notion of a standard model, in turn, may be introduced only after the theory in question has become a fully formalized theory, with overtly specified primitive terms and axioms characterizing them. In this situation the class of all models of the theory in question can be established. This class may consist of only one model or of many models, which depends on the language of the theory and the underlying logic, among other aspects. In the first case we obtain the standard model at once. In the second case we may only choose one of the models and call it standard. I propose to call a model 'standard', if it is most closely related to the intended model. The similarity between intended and standard model should be based on a kind of isomorphism. Because the standard model of a theory is a specific element of the well-defined class of all models of the theory in question, it is a genuine mathematical object and as such it is well-defined, too. We should remember, however, that the name *standard* was given to it on the basis of our decision. The latter was supported by the observed resemblance of the standard model to the intended model given in advance. It may also happen that certain theorems concerning the standard model provide additional support for our decision. Still, the selection of the name *standard* is based primarily on pragmatic criteria.

The standard model of arithmetic is determined uniquely (up to isomorphism) on the basis of second-order Peano axioms. In the case of first-order Peano arithmetic its standard model is only one of the continuum many countable models of this theory. According to Tennenbaum's theorem, it is the only recursive model of this first-order theory. It is also its prime model, meaning that it can be elementarily embedded in any other model of the theory in question. Non-standard models of arithmetic contain infinitely large numbers.

The completely ordered real field (satisfying thus the upper bound property) is determined uniquely (up to isomorphism). It is commonly accepted as the standard model of the arithmetical continuum. It is also a maximal Archimedean field but it is not algebraically closed. The complex field, in turn, is determined uniquely (up to isomorphism) as the only algebraically closed field of the characteristic zero whose transcendence degree over the field of rational numbers equals the continuum. No order compatible with the arithmetical operations is possible in the field of complex numbers.

The (first-order) theory of real closed fields is semantically complete, meaning that all models of this theory are elementarily equivalent, i.e. have the same set of true sentences. The real numbers, which form a real closed field, are thus characterized uniquely with respect to elementary equivalence in the first-order language.

The hyperreal field is also elementarily equivalent with the field of real numbers, but it is not an Archimedean field (it contains infinitesimals). The rather unfortunate name *non-standard analysis* given to the theory concerning the hyperreal field may suggest that hyperreal numbers are non-standard. However, it is mainly the matter of mathematical research practice to decide, on the basis of accumulated knowledge and fruitfulness of applications, which structure should be called standard.

A paper by Solomon Feferman [8] discusses the question of which formal representations of the geometric continuum could be thought of as standard. Feferman lists a few candidates: Euclid's continuum; Cantor's continuum; Dedekind's continuum; Hilbert's continuum; the continuum as the set of all branches in the full binary tree; and the continuum as the family *P(N)* (the full powerset of the set of all natural numbers). Feferman summarizes his paper on conceptions of the continuum as follows:

> Of all the conceptions of the continuum considered here, only those of sec. 3 stand as structural ones, and of those only 2^N and *P(N)* stand as *basic* structural conceptions. For, the continuum in Euclidean and Hilbertian geometry is not an isolated notion, while the continuum as given by Cantor's and Dedekind's construction of the real numbers, are hybrid constructions. The set 2^N of all sequences of 0s and 1s isolates the set-theoretical component of Cantor's construction, while the set *P(N)* of all subsets of *N* isolates that of Dedekind's construction, but both of these lose entirely the basic geometric intuition

> of the continuum. On the other hand, it does not count against Cantor's and Dedekind's conceptions of the continuum in the form of the real number system R that they are hybrids of geometrical, arithmetical and set-theoretic notions. On the contrary, by a kind of miracle of synergy, R has proved to serve together with the natural numbers N as one of the two core structures of mathematics; together they are the *sine qua non* of our subject, both pure and applied.

If first-order Zermelo-Fraenkel set theory is consistent (which cannot be proved in the theory itself), then it has a plentitude of models. It is commonly accepted in the mathematical community to call a model of this theory *standard*, if the interpretation of the membership predicate in it is the real membership relation. Models of set theory without the axiom of foundation are usually seen as non-standard models.

The distinction between *genuine* (*normal, natural*, etc.) mathematical objects and those called *unintended* (*unwilling, imaginary*, etc.) was noticed in the history of mathematics even before the second half of the 19th century. For example, negative or imaginary numbers were long rejected as legitimate mathematical objects before they finally became accepted by the mathematical community. It is important to make a distinction between a *non-standard* (object) and an *innovation*. Haim Gaifman discussed the following *innovations* in mathematics in his paper [11] devoted to the non-standard models: the discovery of irrationals; the incorporation of negative and complex numbers in the numeral system; the extension of the concept of *function* in the nineteenth century; and the discovery of non-Euclidean geometry. Gaifman gives arguments that such innovations should not be considered non-standard. He also discusses certain further candidates for being a standard mathematical object, including *well-ordered* and *constructible* sets. The full powerset operation, on the other hand, escapes from the list of standards.

There are several ways of constructing non-standard models of mathematical theories. Let us consider Peano arithmetic (PA). If we expand its language by a new individual constant c and take into account an infinite set of sentences $C = \{\neg \bar{n} = c : n \in N\}$ (where $\bar{n}$ is the numeral denoting the natural number n), then each of its finite subsets has a model and it follows from the compactness theorem that C itself has a model. The denotation of c in this model is different from each standard natural number and hence the model in question is non-standard. Another possibility, already anticipated by Thoralf Skolem, is to build a suitable ultraproduct (actually, an ultrapower) starting with the standard model of PA. One can also consider a full binary tree of expansions of arithmetic and show that each branch of this tree corresponds to a model of PA; one of them is the standard model, while all others are non-standard models. We will come back to the latter possibility below, discussing Jan Woleński's views on non-standard models.

2. On the Origin of Metalogical Concepts

Claims about uniqueness of models require precise tools of comparison of the models themselves. There are essentially two ways of characterizing the indistinguishability of models of a given theory. One of them is structural: we may ask whether the models are isomorphic (or partially isomorphic, or one of them being a homomorphic image of the other, and so on). The notion of isomorphism emerged in algebraic considerations in the early 19th century. Isomorphic structures are structurally indistinguishable. If all models of a theory T are isomorphic, then we say that T is a *categorical* theory. A theory T is *categorical in power* κ (where κ is an infinite cardinal number), if it has a model of power κ and all its models of power κ are isomorphic. It should be stressed that first-order theories cannot be categorical, with the exception of certain trivial cases. This is a consequence of Löwenheim-Skolem-Tarski theorem which says that if a theory (without finite models) has a model, then it has models of all infinite cardinalities.

Another kind of indistinguishability of models is based on semantic criteria. We say that two models are *elementarily equivalent*, if the sets of sentences true in them coincide. A theory T is (semantically) *complete*, if all its models are elementarily equivalent. If two models are isomorphic, then they are also elementarily equivalent, and hence categoricity implies semantic completeness, but the converse implication does not hold.

The notion of categoricity originated in the papers of Edward Huntington and Oswald Veblen. Huntington used the term *sufficiency* in 1902 and Veblen replaced it by the term *categoricity* in 1904. In the nineteen-twenties Abraham Fraenkel and Rudolf Carnap used the term *monomorphy* (*Monomorphie* in German) in the meaning in question. Fraenkel and Carnap considered also a kind of semantic completeness (called by Carnap *non-forkability*, in German: *nicht-Gabelbarkeit*). It should be stressed that before emergence of well-developed metalogic the notions of categoricity and semantic completeness were not sharply separated. In the absence of precise formal logical tools the claim that isomorphism implies semantic indistinguishability was understood evident by Huntington, Veblen and also earlier by Richard Dedekind. An important early contribution to the relationships between these notions is the paper [15] written by Lindenbaum and Tarski. Tarski's paper [22] from 1940 (printed as appendix in [16]) elaborates further this issue. Tarski introduced the notion of elementary equivalence in the nineteen-fifties. Many important observations concerning the emergence and mutual relations between the notions in question are contained in [1], [6] and [7].

Categoricity, categoricity in power and semantic completeness were further characterized in full detail in classical and modern model theory. There is no need to report on these results here; an interested reader may consult for example [14] or [17]. Let us only add that the tools from model theory are sufficient for talking about several kinds of indistinguishability of models and the uniqueness of these models.

3. Extremal Axioms

The term 'extremal axiom' was introduced in the paper [4] written by Carnap and Bachmann. The authors tried to present a general form of these axioms using the logical framework of the theory of types. At the beginning of the paper they write (citing [5] which is the English translation of [4]):

> Some important axiom systems are so constructed that first a series of axioms is given, making certain statements about the basic concepts of the axiomatic theory, and then at the end an axiom of a special sort appears which apparently speaks about the foregoing axioms and not about the special concepts of the theory. The most famous axiom system of this sort is Hilbert's axiom system of Euclidean Geometry. It ends with the famous 'completeness axiom' which runs as follows [The footnote given here by the authors reads: D. Hilbert, *Grundlagen der Geometrie* (Leipzig and Berlin). We take the Hilbert completeness axiom in the form it has in editions 2–6, not the 'linear formulation' of the 7th edition of 1930. – J.P.]:
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> 'The elements (points, lines, planes) of geometry constitute a system of things which cannot be extended while maintaining simultaneously the cited axioms, i.e., it is not possible to add to this system of points, lines, and planes another system of things such that the system arising from this addition satisfies axioms AI-V1.'
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> Axioms of this sort, which ascribe to the objects of an axiomatic theory a maximal property – in that they assert that there is no more comprehensive system of things that satisfies a given series of axioms – we call a maximal axiom. The same axiomatic role as that of maximal axiom is played in other axiom systems by minimal axioms which ascribe a minimality property to the objects of the discipline. Maximal and minimal axioms we call collectively extremal axioms [5, pp. 68-69].

Besides Hilbert's axiom of completeness in geometry (which was an axiom of maximality) Carnap and Bachmann considered two axioms of minimality: the induction axiom in arithmetic and

Fraenkel's axiom of restriction in set theory. The latter says, roughly speaking, that only these sets exist whose existence can be proved in set theory (and hence the universe of all sets should be as narrow as possible). Extremal axioms were considered by Carnap and Bachmann as expressing a kind of completeness of models and hence as candidates for conditions characterizing models in a unique way. The famous limitative theorems proved later in the 20th century showed the possibilities and restrictions in this respect.

Early Carnap's views on extremal axioms and metalogic are best described in several papers written by Georg Schiemer (see for instance [21]). My book [19] presents logical, mathematical and cognitive aspects of extremal axioms. In particular, I propose to extend the inventory of extremal axioms by taking into account Kurt Gödel's axiom of contructibility, John von Neumann's axiom of the limitation of size and Roman Suszko's axiom of canonicity (these are examples of restriction axioms in set theory, hence axioms of minimality) as well as axioms of the existence of large cardinals in set theory (which are axioms of maximality). I also mention an interesting example of a maximality axiom in algebra, namely a generalization of Dedekind's axiom of continuity proposed by Philip Ehrlich and used by him to prove categoricity results concerning certain non-Archimedean structures.

Hilbert's axiom of completeness in geometry presented in [13] was later replaced by the axiom of continuity for real numbers which resulted, among others, in the proof of categoricity of the system of Euclidean geometry (see for example [3]). Second-order axiom of induction in arithmetic is used in the proof that there exists exactly one (up to isomorphism) Peano algebra. On the other hand, first-order Peano arithmetic is far from being semantically complete (and hence also categorical).

It is interesting that mathematicians have changed their views on extremal axioms in set theory. The axioms of restriction were abandoned, which was most explicitly shown in [10]. Set theoreticians are recently eager to investigate several axioms of the existence of large cardinals which presuppose that the universe of all sets should be as large as possible. Kurt Gödel himself opted for this trend and Ernst Zermelo proposed to accept the existence of the whole transfinite hierarchy of strongly inaccessible numbers already in his second axiomatization of set theory presented in [26].

4. Jan Woleński on Intended and Standard Models

Jan Woleński devoted several works to metatheoretical analysis of formalized theories. In my opinion, most interesting are his proposals involving applications of concepts elaborated in metalogic to the analysis in question. It is justified to claim that Jan Woleński achieved perfection in this work. He may doubtlessly be considered the leading continuator of the famous Warsaw-Lviv school.

We shall analyze in brief Woleński's views on intended and standard models. Our main source is his book on epistemology [25]. Many Polish philosophers wrote on intended models (notably Marian Przełęcki, Adam Nowaczyk, Ryszard Wójcicki, and Adam Grobler) but their analysis was focused mainly on intended models of empirical theories. Jan Woleński's reflections, in turn, are devoted mainly to intended and standard models of mathematical theories which is also the main issue discussed in this note.

Jan Woleński influenced my own views on intended and standard models mainly with respect to the opinion that these models are distinguished not on purely syntactic or semantic criteria but rather by taking into account also certain pragmatic factors. There may be small differences between his understanding of the distinction between intended and standard models and the one presented at the beginning of this note, but they are negligible.

Woleński recalls the construction of the tree of extensions of first-order Peano arithmetic PA ([25], 256; [18], 161). Let T_0=PA and let ψ_0 be any undecidable statement in T_0. We put: T_{00} = PA + ψ_0 and T_{01} = PA + $\neg\psi_0$. For any finite 0–1 sequence σ let: $T_{\sigma 0} = T_\sigma + \psi_\sigma$ and $T_{\sigma 1} = T_\sigma + \neg\psi_\sigma$, where ψ_σ is any undecidable sentence of T_σ (for any T_σ there exists such an undecidable sentence).

We obtain in this way the full binary tree of extensions of PA. This tree has continuum many branches. It follows from the compactness theorem that the union of theories from each branch is consistent (under the assumption of consistency of PA) and hence each such union has a model. Further, due to the downward Löwenheim-Skolem theorem each such union has a countable model. No two such models are elementarily equivalent which follows from the construction of the above tree. Consequently, no two such models are isomorphic.

Let ψ_0 be identical with *Con(PA)* (that is, the sentence expressing the fact that PA is consistent) and let ψ_α express the consistency of T_α. Then the model of the leftmost branch of the above tree is isomorphic to the standard model of PA. All other branches have countable non-standard models. Each sentence of the form $\neg Con(T_\alpha)$ has the Gödel number which is a non-standard natural number in the respective model. Let us note on the margins that PA is a *wild* theory: it has, in each infinite power κ, the maximum possible number of models, that is 2^κ (provided the consistency of PA, of course).

The standard countable model of PA can be distinguished out of the totality of countable models of this theory only using some metatheoretical results, as already mentioned above. However, Jan Woleński proposes a more deep and subtle analysis of this issue. We need some auxiliary tools to present his views here:

A theory T is *descriptively complete* (in short: *o-complete*) with respect to a sequence $(a_s)_{s\in S}$ of individual constants (where S is any index set), if for any formula $\varphi(x)$ of the language of T with one free variable x the following implication holds: if $\varphi(x/a_s)$ is a theorem of T for all $s \in S$, then also $\forall x\varphi(x)$ is a theorem of T. If the sequence of individual constants in question is countable, then we say that T is *ω-complete*.

A theory T is *constructive* with respect to a sequence of terms $(t_s)_{s\in S}$, if for any formula $\varphi(x)$ of the language of T with one free variable x the following implication holds: if $\exists x\varphi(x)$ is a theorem of T, then $\varphi(x/t_s)$ is a theorem of T for some $s \in S$.

A theory T is *o-consistent* with respect to a sequence of terms $(t_s)_{s\in S}$, if for any formula $\varphi(x)$ of the language of T with one free variable x the following implication holds: if $\varphi(x/t_s)$ is a theorem of T for all $s \in S$, then $\exists x\neg\varphi(x)$ is not a theorem of T. If the sequence of terms in question is countable, then we say that T is *ω-consistent*. If a theory T is not ω-consistent, then we say that T is *ω-inconsistent*.

By the *ω-rule* we understand a rule of inference with an infinite set of premisses $\varphi(0), \varphi(\overline{1}), \varphi(\overline{2}), \ldots$ and the conclusion $\forall x\varphi(x)$.

These notions are related to the possibility of associating names with the elements of the domain of a model. ω-consistency was used already by Kurt Gödel in the formulation of his first incompleteness theorem. Descriptive completeness and constructivity were used by Andrzej Grzegorczyk in his famous paper on categoricity [12]. If the language of our theory contains numerals, then we can talk in this language about specific natural numbers. There arises a question of how these properties can be used in the characterization of models of a theory.

For any model $\boldsymbol{M}$ let $Th(\boldsymbol{M})$ denote the *theory of* $\boldsymbol{M}$, that is the set of all sentences true in $\boldsymbol{M}$. Let $\boldsymbol{N}_0$ denote the standard model of PA, $\boldsymbol{N}_c$ the non-standard model obtained by using the compactness argument in the way described above and $\boldsymbol{N}_{in}$ the non-standard model of the theory $PA+ \neg Con(PA)$ obtained from the tree of expansions of PA presented earlier. The set $Th(\boldsymbol{N}_0)$ is thus the set of all arithmetical truths, that is true sentences about standard natural numbers. We recall that PA is incomplete and essentially undecidable. It is not finitely axiomatizable. If we add the infinitary ω-rule to PA, then the enriched theory becomes complete, but the price for that is very high, because we admit infinitary proofs, which is of course a debatable decision.

Jan Woleński uses an original generalization of the traditional square of oppositions for a formal representation of the logical dependencies between the notions of consistency, inconsistency, ω-consistency, and ω-inconsistency. It should be noted that these generalizations (see [24]) appeared to be a very productive and effective tool of logical analysis as shown by Woleński in his numerous articles on analytical philosophy. We are interested here mainly in possibilities of applying the notions in question to the characterization of intended and standard models.

All axioms and theorems of PA are true in the model $\boldsymbol{N}_{in}$. However, the sentence $\neg Con(PA)$ is also true in $\boldsymbol{N}_{in}$. The Gödel number of this sentence cannot be a standard natural number because otherwise PA would prove its own inconsistency, contrary to what was assumed. The sentence $\neg Con(PA)$ is obviously false in the standard model $\boldsymbol{N}_0$ and Woleński writes that it is difficult to express its sense in the language appropriate for talking about $\boldsymbol{N}_0$. If we are looking for formal criteria of being the standard model of arithmetic, then a good candidate could be the well-ordering property of the set of natural numbers. Woleński shares this opinion with Haim Gaifman (see [11]).

The set $Th(\boldsymbol{N}_0)$ of all standard arithmetical truths is ω-consistent, ω-complete and constructive with respect to the sequence of all numerals. Woleński argues that o-consistency and constructivity are too strong conditions for the characterization of an arbitrary set of true sentences. For example, the set $Th(\boldsymbol{N}_{in})$ is consistent but ω-inconsistent. It cannot be constructive, because consistency and constructivity imply ω-consistency. Further, Woleński adds that it is possible to consider the set $Th(\boldsymbol{N}_c)$ as o-consistent and constructive with respect to a suitably chosen sequence of constants. Then $Th(\boldsymbol{N}_c)$ is also o-complete. Woleński concludes from this that consistency (even maximal consistency) and o-completeness are minimal syntactic conditions characterizing the set of sentences true in any model and that the existence of theories which are consistent but at the same time ω-inconsistent clearly shows that truth differs essentially from provability. The semantic theory of truth alone is unable to distinguish the standard model in the class of all models.

Woleński says a few words explicating the commonly accepted assumption that PA is (a formal representation) of the True Arithmetic. From the point of view of a mathematician this could mean that the True Arithmetic is simply the totality of all logical consequences of the axioms of PA, even if not all of them have real applications. Another position (taken by a logician, according to Woleński) could accept the set $Th(\boldsymbol{N}_0)$ as the True Arithmetic, thus identifying it with all arithmetical truths. Non-standard models of arithmetic can nevertheless be fundamental in certain mathematical disciplines – a notable example is the hyperreal field which has become recently more and more important in mathematical analysis.

Woleński expresses a few interesting remarks concerning the ways of formalization of arithmetic. The class of models isomorphic to $\boldsymbol{N}_0$ can be characterized in second-order logic and this fact is considered a virtue of such formalization, first of all by the professional mathematicians. However, second-order arithmetic is undecidable and incomplete. The great expressive power of second-order logic is related to the acceptance of the absolute notion of a set. The expressive power of a logic is inversely proportional to its deductive power. Jan Woleński explicitly opts for first-order formalization, which possesses a lot of 'good' deductive properties and adds that this choice does not have any influence on the criteria of standardness of models.

The monograph [25] contains a very detailed analysis of the notion of an *analytic sentence*. One type of such sentences is relevant to standard models. Woleński proposes to call a sentence ψ *analytical in the pragmatic sense*, if there exists a theory T such that ψ is a theorem of T and ψ is true in the intended model of T. From the formal (logical) point of view standard models are as good as non-standard ones. It is our epistemic decision to call a model standard. We have argued in the first part of this note that this decision is determined by reflecting on the properties of the intended model, a structure investigated prior to the emergence of the formal (axiomatic) theory.

The monograph in question contains also a critique of Putnam's arguments expressed in [20]. Jan Woleński shows that Putnam is wrong claiming that models are nothing else but constructions inside theories. Putnam assumes that we refer to models (in particular to the intended model) always using the tools of the corresponding theory. This is clearly false, writes Woleński, because we must refer to metatheory when distinguishing between models. This is obvious for instance in the explication of Skolem's paradox in the context of models of the theory of real numbers. We switch to metatheory asserting that the proper (adequate) model of this theory has a power of continuum. The impossibility of definition of models in the object language, which follows from metalogical results, is discussed in more detail in [23].

5. Concluding Remarks

The main goal of this note was to present Jan Woleński's views on intended and standard models of mathematical theories. His contribution to this issue is based on an original application of metalogical results to philosophical problems. One can hardly find in philosophical literature examples of formal analysis comparable in depth and subtlety to those provided by Jan Woleński. My own distinction between intended and standard models was influenced by his proposals. In a sense, the distinction in question slightly resembles the distinction between the intuitive notion of a computable function and any precise mathematical representation of computability (for instance recursive functions or Turing machines).

Woleński's remarks are related first of all to models of arithmetic and to a lesser extent to geometric continuum and set theory. Taking into account the history of mathematics on a large timeline it seems legitimate to say that the intended model of arithmetic is much better understood than the continuum. The long philosophical debate about the structure of a continuum is still vivid and far from ultimate conclusions. The most commonly accepted representation of the geometric continuum by the arithmetical continuum of real numbers competes with the quite new representation based on hyperreal numbers. One can also find the opinion that the continuum should not be considered as a set of points, though no well-developed mathematically correct alternative is in sight at the present moment. This situation may prompt us to the conclusion that mathematicians have described several aspects of the continuum but have not captured ***the*** intended model of the continuum yet. A very interesting recent review of opinions on the structure of the continuum can be found in [2]. The discussion concerning models of set theory is also far from being closed as is clearly visible from the research directed towards new axioms which could characterize the set-theoretical universe in a more unique way.

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