Studia Humana
Volume 9:3/4 (2020), pp. 10—18
DOI: 10.2478/sh-2020-0025

Proof *vs* Truth in Mathematics

Roman Murawski

Adam Mickiewicz University
Uniwersytetu Poznańskiego 4 Street
61-614 Poznań, Poland

e-mail: rmur@amu.edu.pl

Abstract:
Two crucial concepts of the methodology and philosophy of mathematics are considered: proof and truth. We distinguish between informal proofs constructed by mathematicians in their research practice and formal proofs as defined in the foundations of mathematics (in metamathematics). Their role, features and interconnections are discussed. They are confronted with the concept of truth in mathematics. Relations between proofs and truth are analysed.
Keywords: formal proof, informal proof, truth, mathematics, logic, incompleteness, Jan Woleński.

1. Introduction

Concepts of proof and truth play an important role in metamathematics, especially in the methodology and the foundations of mathematics. Proofs form the main method of justifying mathematical statements. Only statements that have been proved are treated as belonging to the corpus of mathematical knowledge. Proofs are used to convince the readers of the truth of presented theorems. But what is a proof? What does it mean in mathematics that a given statement is true? What is truth (in mathematics)?

In mathematical research practice proof is a sequence of arguments that should demonstrate the truth of the claim. Of course, particular arguments used in a proof depend on the situation, on the audience, on the type of a claim, etc. Hence the concept of proof has in fact a cultural, psychological and historical character. In practice mathematicians generally agree on whether a given argumentation is a proof. More difficult is the task of defining a proof as such. Beside the concept of proof used in research practice there is a concept of proof developed by logic. What are the relations between those two concepts? What roles do they play in mathematics?

On the other hand the concept of truth belongs to the fundamental concepts that have been considered in epistemology since ancient Greece.[1] There were many attempts to define this vague concept. The classical definition (attribued to Aristotle) says that a statement is true if and only if it agrees with the reality, or – as Thomas Aquinas put it: "Veritas est adequatio intellectus et rei, secundum quod intellectus dicit esse quad est vel non esse quod non est" (*De veritate*, 1, 2).

But what does it mean that a mathematical statement (for example: "2 + 2 = 4") agrees with the reality? With what reality? One can answer: "With mathematical reality?" But what is mathematical reality? And we come here to one of the fundamental problems of the ontology of mathematics: where and how do mathematical objects exist? Is the mathematical universe a reality or an artifact?

2. Proof in Mathematics: Formal *vs* Informal

Mathematics was and still is developed in an informal way using intuition and heuristic reasonings – it is still developed in fact in the spirit of Euclid (or sometimes of Archimedes) in a *quasi*-axiomatic way. Moreover, informal reasonings appear not only in the context of discovery but also in the context of justification. Any correct methods are allowed to justify statements. Which methods are correct is decided in practice by the community of mathematicians. The ultimate aim of mathematics is "to provide correct proofs of true theorems" [2, p. 105]. In their research practice mathematicians usually do not distinguish concepts "true" and "provable" and often replace them by each other. Mathematicians used to say that a given theorem holds or that it is true and not that it is provable in such and such theory. It should be added that axioms of theories being developed are not always precisely formulated and admissible methods are not precisely described.[2]

Informal proofs used in mathematical research practice play various roles. One can distinguish among others the following roles (cf. [4], [7]):

(1) verification,
(2) explanation,
(3) systematization,
(4) discovery,
(5) intellectual challenge,
(6) communication,
(7) justification of definitions.

The most important and familiar to mathematicians is the first role. In fact only verified statements can be accepted. On the other hand a proof should not only provide a verification of a theorem but it should also explain why does it hold. Therefore mathematicians are often not satisfied by a given proof but are looking for new proofs which would have more explanatory power. Note that a proof that verifies a theorem does not have to explain why it holds. It is also worth distinguishing between proofs that convince and proofs that explain. The former should show that a statement holds or is true and can be accepted, the latter – why it is so. Of course there are proofs that both convince and explain. The explanatory proof should give an insight in the matter whereas the convincing one should be concise or general. Another distinction that can be made is the distinction between explanation and understanding. In the research practice of mathematicians simplicity is often treated as a characteristic feature of understanding. Therefore, as G.-C. Rota writes: "[i]t is an article of faith among mathematicians that after a new theorem is discovered, other, simpler proof of it will be given until a definitive proof is found" [23, p. 192].

It is also worth quoting in this context Aschbacher who wrote:

> The first proof of a theorem is usually relatively complicated and unpleasant. But if the result is sufficiently important, new approaches replace and refine the original proof, usually by embedding it in a more sophisticated conceptual context, until the theorem eventually comes to be viewed as an obvious corollary

> of a larger theoretical construct. Thus proofs are a means for establishing what is real and what is not, but also a vehicle for arriving at a deeper understanding of mathematical reality [1, p. 2403].

As indicated above a concept of a "normal" proof used by mathematicians in their research practice (we called it "informal" proofs) is in fact vague and not precise. In the 19th century there appeared a new trend in the philosophy of mathematics and in the foundations of it whose aim was the clarification of basic mathematical concepts, especially those of analysis (cf. works by Cauchy, Weierstrass, Bolzano, Dedekind). One of the drivers of this trend was the discovery of antinomies in set theory (due among others to C. Burali-Forte, G. Cantor, B. Russell) and of semantical antinomies (among others by G. D. Berry and K. Grelling). All those facts forced the revision of fundametal concepts of metamathematics.

One of the formulated proposals was the programme of David Hilbert and the formalism based on it. Hilbert's main aim was to justify mathematics developed so far, in particular to show that mathematics using the concept of an actual infinity is consistent and secure. To achieve this aim Hilbert proposed to develop a new theory called proof theory (*Beweistheorie*). It should be a study of proofs in mathematics – however not of real proofs constructed by mathematicians but of formal proofs. The latter played a fundamental role in Hilbert's programme. Hilbert proposed to formalize all theories of the entirety of mathematics and to prove the consistency of them. Note that he did not want to replace the mathematics developed by mathematicians by formalized theories – the formalization was for him only a methodological tool that should enable the study of theories as such.

To formalize a theory one should first fix a symbolic formal language with formal rules of constructing formulas in it, then fix appropriate axioms expressed in this language as well as accepted rules of inference which again should have an entirely formal and syntactic character. A proof (exactly: formal proof) of a formula φ in such a theory is now a sequence of formulas $\varphi_1, \varphi_2, \dots, \varphi_n$ such that the last member of the sequence is the formula φ and all members of it either belong to the set of presumed axioms or are consequences of previous members of the sequence according to one of the accepted rules of inference. Observe that this concept of a formal proof has a syntactic character and does not refer to any semantical notions such as meaning or interpretation.

Note that formalization is connected also with the idea of mathematical rigor. Detlefsen [6, p. viii] writes:

> [W]ith the vigorous development of techniques of *formalization* that has taken place in this [i.e., 20th century – my remark, R.M.] century, demands for rigor have increased to a point where it is now the reigning orthodoxy to require that, to be genuine, a proof must be formalizable. This emphasis on formalization is based on the belief that the only kinds of inferences ultimately to be admitted into mathematical reasoning are *logical* inferences [. . .].

Comparing the usual proofs of mathematical research practice (informal proofs) and formal proofs one can see that both types of proofs consist of steps of deduction. They differ by the properties of those steps. According to Hamami [10] one can distinguish here three types of differences: formality, generality and mechanicality. Informal inferences are meaning dependent, matter dependent and content dependent wheras formal inferences are meaning, matter and content independent. Hamami [10, p. 679] writes: "To say that logical inference is *formal* is to say that it is governed by rules of inference which only depend on the logical form of premisses and conclusion, and not on their meaning, matter, or content."

Tarski [25, p. 187] said: "[T]he relation of following logically is completely independent of the sense of the extra-logical constants occurring in the sentences among which this relation obtains […]."

Informal inferences are non-general wheras formal ones are general. This means in particular that the former are topic-specific, subject matter dependent and domain dependent, and the latter are topic-neutral, subject matter and domain independent. Detlefsen [5, p. 350] wrote in connection with this:

> The mathematician's inferences stem from and reflect a knowledge of the local "architecture" (Poincaré's term) of the particular subject with which they are concerned, while those of the logician represent only a globally valid, topic-neutral (and, therefore, locally insensitive!) form of knowledge.

Hamimi [10] explains that the claim that logical inference is general means in particular that "it is governed by rules of inference that are *generally applicable*, i.e., that are applicable to propositions – premisses and conclusions – belonging to any and every topic, subject matter, or domain" [10, pp. 684-685].

The last difference between informal and formal proofs distinguished by Hamimi is the property of mechanicality: informal ones are non-mechanical and formal ones – mechanical. What does it mean is explained by the following quotations. Kreisel [16, p. 21] writes:

> Mathematical reasoning, except in the 'limiting' case of numerical computations, does not present itself to us as the execution of mechanical rules [. . .] The connection between reliability and the possibility of mechanical checking is usually, and somewhat uncritically, taken for granted.

And Hamimi [10, p. 695] says: "To say that logical inference is *mechanical* is to say that it is governed by rules of inference that are mechanical."

One can distinguish here two senses in which logical rules of inference are mechanical: mechanical applicability and mechanical checkability.

Add at the end of this section that the concept of a formal proof enables us to study mathematical theories as theories, to investigate their properties, etc. It makes possible the entirety of metamathematics. However, the following question arises: what are the relations between formal and informal proofs. Recall that the first one is a practical notion of a semantical character, not having a precise definition. The latter is a theoretical concept of a syntactical character used in logical studies. Mathematicians are usually convinced that every "normal", i.e., informal mathematical proof can be transformed into a formalized one, however there are no general rules describing how this can and should be done. This thesis is sometimes called Hilbert's thesis. Barwise [3] wrote:[3] "[T]he informal notion of provable used in mathematics is made precise by the formal notion *provable in first-order logic*. Following a sug[g]estion of Martin Davis, we refer to this view as *Hilbert's Thesis*."

In fact a formalization of an informal proof requires often some original and not so obvious ideas.

3. Truth in Mathematics

We indicated above that "normal" mathematicians (i.e., mathematicians not being logicians or specialists in the foundations of mathematics) do not distinguish in their research practice between provability (in the broad sense) and truth. Moreover, those two concepts are usually identified in practice. This was done also by formalists.[4] Gödel wrote in a letter of 7th March 1968 to Hao Wang [cf. 29, p. 10]: "[...] formalists considered formal demonstrability to be an *analysis* of the concept of mathematical truth and, therefore were of course not in a position to *distinguish* the two."

Note that "mathematical truth" should be understood here in an intuitive way. Moreover, the informal concept of truth was not commonly accepted as a definite mathematical notion in Hilbert's and Gödel's time. There was also no definite distinction between syntax and semantics. This explains also, in some sense, why Hilbert preferred to deal in his metamathematics solely with forms of formulas, using only finitary reasonings which were considered to be secure – contrary to semantical reasonings which were non-finitary (sometimes called: infinitary) and consequently not secure.

The precise definition of truth was given by Tarski in his famous paper *Pojęcie prawdy w językach nauk dedukcyjnych* [24]. Referring to the classical Aristotle's definition he attempted to make more precise the concept of truth with respect to formalized languages. In such languages "the sense of every expression is unambiguously determined by its form" [27, p. 186].

Tarski defined the concept of truth by using the concept of satisfaction, more exactly, satisfaction of a formula on a valuation by a given interpretation of primitive notions of the considered language, hence in a given structure. His definition refers to the so called convention (T) according to which the statement "Snow is white" is true if and only if snow is white. In fact Tarski did not give a definition of truth but defined only the class of true sentences (of a given language).

Tarski's definition has an infinitary character – the infinity appears in the reference to infinite sequences of elements of the considered structure (valuations) as well as in the case of satisfaction of formulas with quantifiers. It does not go beyond the extensional adequacy and does not explain the essence of the truth and of being true. It relativizes also the concept of truth to a given structure or domain.

In the above mentioned paper [24] Tarski formulated also the theorem on the undefinability of truth. It says that the conccept of truth for given formalized language cannot be definied in this language itself – to do this more powerful means are necessary. In other words: the set of sentences true in a given structure is not definable in it (though in some cases it is definable with parameters). Tarski formulated this theorem as Theorem I, point (β) [cf. 26, p. 247]:[5] "[A]ssuming that the class of all provable sentences of the metatheory is consistent, it is impossible to construct an adequate definition of truth in the sense of convention T on the basis of the metatheory."

One of the consequences of Tarski's theorem is the fact that in order to construct truth theory, for example, for the language of the arithmetic of natural numbers (hence a theory of finite entities) one should apply more powerful means, in fact the infinity. In other words: the concept of an arithmetical truth is not arithmetically definable. Generally: semantics needs the infinity! It indicates also the gap between the syntactical concept of a (formal) proof and (formal) provability on the one side and the concept of truth. In fact, for example, the set of true arithmetical sentences is not definable in the language of arithmetics whereas the set of provable sentences (theorems) of arithmetic is arithmetically definable, even more: it is definable by a simple formula (more exactly: by a formula with one existential quantifier and logical connectives as well as eventually bounded quantifiers). Hence one can say that the concept of truth transcends all syntactical means.

The indicated difference between the (definablity of the concept of) provability and (the undefinability of the concept of) truth was the key reason for the famous incompleteness theorems proved by Gödel [8]. Gödel wrote on his discovery in a draft reply to a letter dated 27th May 1970 from Yossef Balas, then a student at the University of Northern Iowa [30, pp.

84-85] and indicated there that it was precisely his recognition of the contrast between the formal definability of provability and the formal undefinability of truth that led him to his discovery of incompleteness. The first incompleteness theorem implies that in every consistent theory containing the arithmetic of natural numbers there are undecidable (i.e. that can neither be proved nor disproved) statements φ such that one formula of the pair φ and *non*-φ is satisfied/true in the intended (standard) model of the theory. It shows that (formal) provability is not the same as truth! However both these concepts are connected by the completeness theorem stating that a statement φ is a theorem of a theory *T* if and only if φ is true in *every* model of *T*. And theories usually possess (infinitely) many various models – not only the intended one (called: standard). So we have that:

1. if a formula φ is provable in the theory *T* then it is true in every model of *T*, hence also in the intended model of *T*,
2. it is not true that for any formula φ: if φ is true in the intended (standard) model of *T* then it is provable in *T*.

Add that when "normal" mathematicians are saying that a given sentence φ is true then they have in mind that it is true in the intended (standard) model.

One should mention also another phenomenon. As indicated above the concept of truth/true sentence for a given language *L* is not definable in the language *L* itself. However partial concepts of truth for formulas of *L* are definable in *L*. More exactly: if one considers only formulas of *L* with a given maximal number of quantifiers (this is in fact a restiction of the complexity of a formula) then the concepts of satisfaction and truth for such formulas of a language *L* are definable in *L*. It can be proved that the definition of the satisfaction predicate for formulas with maximally *k* quantifiers is a formula with *k* quantifiers, i.e., a formula of the same degree of complexity. Details can be found in our monograph [18].

The concept of truth/true formula can be investigated also by mathematical, more exactly: by axiomatic-deductive methods. Conditions formulated in Tarski's definition of truth can be treated as axioms characterizing the predicate of being satisfied and true. Such an approach has been studied in detail for the case of arithmetic of natural numbers – cf. for example [17] and [21].

Results obtained by described investigations show that not for every model of arithmetic one can define a concept of satisfaction and truth on it having natural properties assumed and required by Tarski's definition. A necessary condition is here the property that the model should be recursively saturated.[6] Additional properties of a model must be assumed if one requires that the concept of truth upon a given model have some useful (and natural) properties like being full (i.e., deciding the truth of every formula on any valuation) or being inductive (this property means that the induction principle holds not only with respect to formulas of the language of arithmetic but also for an extended language augmented by the satisfaction/truth predicate).

It also turns out that if a concept of satisfaction and truth (called a satisfaction class[7]) for a given structure can be defined then it can be done in many mutually inconsistent ways, i.e., if there exists a satisfaction class on the model then there exist many such satisfaction classes. This shows that the axiomatic characterization of the concept of satisfaction and truth based on Tarski's definition is not complete and unique, that Tarski's conditions (treated as axioms) are too weak. This phenomenon can be removed by allowing more powerful – for example set-theoretical – means. All this shows the complexity of the concept of truth.

We indicated above the gap between provability and syntactical concepts on the one hand and satisfaction/truth and semantical concepts on the other. However it turns out that the concept of truth can be (in a certain sense) replaced by the concept of consistency (hence:

a syntactical concept) in the so called ω-logic (it is a generalization of the usual classical logic obtained by admitting the so called ω-rule and reasonings of infinite length) and by the transfinite induction.[8] This confirms the thesis that semantical concepts such as satisfaction and truth require infinitary means. Such concepts can be expressed or replaced by richer syntactical ones, however, this requires the resignation from the requirement of being finitary, in particular from the natural requirement that a proof must have a finite length and can refer only to finitely many assumptions.

4. Conclusion

In research practice mathematicians do not fix and do not restrict allowed methods of proof – any correct method is practically allowed. A mathematician wants to know what properties the considered and investigated structure (intended structure/model, standard structure/model) has or whether a particular property is true/holds in this structure. She/he is not interested in the problem of whether this property can be deduced from a certain given and restricted set of axioms. Therefore, for example, a specialist in number theory who investigates the structure of the natural numbers (i.e., the structure $(\mathbf{N}, S, +, \cdot, 0)$ where $\mathbf{N}$ is the set $\{0, 1, 2, 3, \ldots\}$, S denotes the successor function, $+$ and $\cdot$ denote, resp., addition and multiplication of natural numebrs and 0 denotes the distinguished element called "zero") is not working in the framework of a fixed axiomatized formal system of arithmetic but is using any correct mathematical methods in order to decide whether a considered property is true/holds in the investigated structure (in the intended, standard model of arithmetic of natural numbers). Consequently she/he does not hesitate to use even methods of complex analysis (as is done in the analytic numer theory) if only they can be useful in deciding the considered problem.

The informal and vague concept of proof used by mathematicians in their research practice can be made precise by the concept of formal proof. The latter makes possible exact metamathematical investigations of mathematical theories – more exactly of their formal counterparts (and not of real theories considered by "normal" mathematicians). However the formal concept of proof (with precisely described and restricted rules of inference) as well as the very concept of formalized theory based on it have some limitations indicated by Gödel's incompleteness theorems. On the other hand the precise concept of satisfaction and truth relativizes truth to a given structure/interpretation. The concept of formal proof is adequate with respect to *all* models of a considered theory (as the completeness theorem states) and not only to the truth in the intended/standard structure. All this implies that metamathematical studies of proofs, structures, theorems and theories are not exact counterparts of what mathematicians are really doing in their research practice, they are in fact idealizations of the real practice.

Let us finish our considerations by quating Alfred Tarski who in the paper "Truth and proof" wrote:

> Proof is still the only method used to ascertain the truth of sentences within any specific mathematical theory. […] The notion of a true sentence functions thus as an ideal limit which can never be reached but which we try to approximate by gradually widening the set of provable sentences. […] There is no conflict between the notions of truth and proof in the development of mathematics; the two notions are not at war but live in peaceful coexistence [27, p. 77].

References

1. Aschbacher, M. Highly complex proofs and implications of such proofs. *Philosophical Transactions of the Royal Society (A)* 363, 2005, pp. 2401–2406.
2. Avigad, J. Mathematical method and proof, *Synthese* 153, 2006, pp. 105–159.
3. Barwise, J. An introduction to first-order logic, In J. Barwise (ed.), *Handbook of Mathematical Logic*, Amsterdam: North-Holland, 1977, pp. 5–46.
4. CadwalladerOlsker, T. What do *we* mean by mathematical proof? *Journal of Humanistic Mathematics* 1, 2011, pp. 33–60.
5. Detlefsen, M. Poincaré against the logicians, *Synthese* 90 (3), 1992, pp. 349–378.
6. Detlefsen, M. (ed.). *Proof, Logic and Formalization*, London: Routledge, 1992,
7. De Villiers, M. D. *Rethinking Proof with the Geometer's Sketchpad*, Emeryville, CA: Key Curriculum Press, 1999.
8. Gödel, K. Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme. I, Monatshefte für Mathematik und Physik 38, 1931, pp. 173-198. Reprinted with English translation: On formally undecidable propositions of *Principia Mathematica* and related systems, In Gödel K. *Collected Works*, vol. I: *Publications 1929-1936*, S. Feferman, J. W. Dawson, Jr., S. C. Kleene, G. H. Moore, R. M. Solovay and J. van Heijenoort eds.), New York: Oxford University Press, and Oxford: Clarendon Press, pp. 144–195.
9. Gödel K. *Collected Works*, vol. I: *Publications 1929-1936*, S. Feferman, J. W. Dawson, Jr., S. C. Kleene, G. H. Moore, R. M. Solovay and J. van Heijenoort eds.), New York: Oxford University Press, and Oxford: Clarendon Press.
10. Hamami, Y. Mathematical inference and logical inference, *The Review of Symbolic Logic* 11 (4), 2019, pp. 665–704.
11. Kahle, R. Is there a "Hilbert thesis"? *Studia Logica* 107, 2019, pp. 145–165.
12. Kaye, R. *Models of Peano Arithmetic*, Oxford: Clarendon Press, 1991.
13. Kotlarski, H., and Z. Ratajczyk. Inductive full saisfaction classes, *Annals of Pure and Applied Logic* 47, 1990, pp. 199-223.
14. Kotlarski, H., and Z. Ratajczyk. More on induction in the language with a full satisfaction class, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 36, 1990, pp. 441-454.
15. Krajewski, S. Non-standard satisfaction classes, In W. Marek, M. Srebrny and A. Zarach (eds.), *Set Theory and Hierarchy Theory*, Proc. Bierutowice Conf. 1975, Lecture Notes in Mathematics 537, Berlin-Heidelberg-New York: Springer Verlag, 1976, pp. 121-144.
16. Kreisel, G. The formalist-positivist doctrine of mathematical precision in the light of experience, *L'Âge de la Science* 3, 1970, pp. 17–46.
17. Murawski R. Satisfaction classes – a survey, In R. Murawski and J. Pogonowski (eds.), *Euphony and Logos*, Amsterdam/Ątlanta, GA: Edition Rodopi, 1997, pp. 259–281.
18. Murawski, R. *Recursive Functions and Metamathematics. Problems of Completeness and Decidability, Gödel's Theorems*, Dordrecht/Boston/London: Kluwer Academic Publishers, 1999.
19. Murawski, R. Truth vs. provability – philosophical and historical remarks. *Logic and Logical Philosophy* 10, 2002, pp. 93–117.
20. Murawski, R. On the distinction proof-truth in mathematics, In P. Gärdenfors et al. (eds.), *In the Scope of Logic, Methodology and Philosophy of Science*, Dordrecht–Boston–London: Kluwer Academic, 2002, pp. 287–303.
21. Murawski, R. Troubles with (the concept of) truth in mathematics, *Logic and Logical Philosophy* 15, 2006, pp. 285–303. Reprinted in: R. Murawski, *Lógos and Máthēma. Studies in the Philosophy of Mathematics and History of Logic*, Frankfurt am Main: Peter Lang International Verlag der Wissenschaften, 2011, pp. 187–201.

22. Murawski, R. Some historical, philosophical and methodological remarks on proof in mathematics, In D. Probst and P. Schuster (Eds.), *Concepts of Proof in Mathematics, Philosophy, and Computer Science*, Ontos Mathematical Logic, Berlin: Walter de Gruyter, 2016, pp. 251–268.
23. Rota, G.–C. The phenomenology of mathematical proof, *Synthese* 111, 1997, pp. 183–196.
24. Tarski, A. *Pojęcie prawdy w językach nauk dedukcyjnych*, Warszawa: Towarzystwo Naukowe Warszawskie, 1933, Wydział III Nauk Matematyczno–Fizycznych, vol. 34. Reprinted in: A. Tarski, *Pisma logiczno-filozoficzne*, vol. 1: *Prawda*, Warszawa: Wydawnictwo Naukowe PWN, 1995, pp. 131–172. English translation: The concept of truth in formalized languages, In A. Tarski, *Logic, Semantics, Metamathematics. Papers from 1923 to 1938,* Oxford: Clarendon Press, 1956, pp. 152–278 and in A. Tarski, *Logic, Semantics, Metamathematics. Papers from 1923 to 1938,* second edition edited and introduced by J. Corcoran, Indianapolis: Hackett Publishing Co., 1983, pp. 152–283.
25. Tarski, A. On the concept of following logically, *History and Philosophy of Logic* 23, 1936/2002, pp. 155–196.
26. Tarski, A. *Logic, Semantics, Metamathematics. Papers from 1923 to 1938*, Oxford: Clarendon Press, 1956.
27. Tarski, A. Truth and proof, *Scientific American* 220, No. 6, 1969, pp. 63–77.
28. Tarski, A. *Logic, Semantics, Metamathematics. Papers from 1923 to 1938,* second edition edited and introduced by J. Corcoran, Indianapolis: Hackett Publishing Co., 1983.
29. Wang, H. *From Mathematics to Philosophy*, London: Routledge and Kegan Paul, 1974.
30. Wang, H. *Reflections on Kurt Gödel*, Cambridge, Mass: The MIT Press, 1987.
31. Woleński, J. *Semantics and Truth*, Logic, Epistemology and the Unity of Science 45, Berlin: Springer Verlag, 2019.

Notes

1. For the development of the concept of truth see Woleński [31]
2. For more on proofs in mathematics and their role see, for example, Murawski [22].
3. Cf. Kahle [11].
4. For the development of the process of distinguishing concepts of provability and truth see, for example, Murawski [19] and [20].
5. Add that in the footnote Tarski explicitly states that his proof of this theorem uses Gödel's method of arithemtization of syntax and his method of diagonalization, however he stresses that he obtained his result independently.
6. For definition see for example Kaye [12].
7. The concept of a satisfaction class was introduced in Krajewski [15] and studied among others by Roman Kossak, Henryk Kotlarski, Stanisław Krajewski, Alistair Lachlan, Roman Murawski, Zygmunt Ratajczyk.
8. Cf. Kotlarski and Ratajczyk [13] as well as [14].