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Characterising Context-Independent Quantifiers and Inferences

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Abstract:
Context is essential in virtually all human activities. Yet some logical notions seem to be context-free. For example, the nature of the universal quantifier, the very meaning of “all”, seems to be independent of the context. At the same time, there are many quantifier expressions, and some are context-independent, while others are not. Similarly, purely logical consequence seems to be context-independent. Yet often we encounter strong inferences, good enough for practical purposes, but not valid. The two types of examples suggest a general problem: how to characterise the context-free logical concepts in their natural environment, that is, in the field of their context-dependent associates. A general Thesis on Quantifiers is formulated: among all quantifiers, the context-free ones are just those definable by the universal quantifier. The issue of inferences is treated following the approach introduced by Richard L. Epstein: valid ones are an extreme case, the result of the disappearance of context-dependence. This idea can be applied to an analysis of a form of abduction, called “reductive inference” in Polish literature on logic. A tentative Thesis on Inferences identifies the validity of a strong inference that is context-independent.
Keywords: philosophical logic, context-independence, context-dependence, quantifier, generalised quantifier, inference, validity, abduction.

1. Introduction

Context is essential in virtually all human activities. If there are exceptions, it seems that they can be found most easily in logic. Some logical notions seem context-free. The matter is not that simple, however, because each application of a notion, even a logical one, depends on the context of the application. For example, the universal quantifier refers to all elements of either an explicitly defined or intended domain. The domain constitutes its context. However, the nature of the universal quantifier, the very meaning of “all”, seems to be independent of the context. Similarly, all real-life inferences and also mathematical proofs — especially proofs within living mathematics as opposed to official presentation of results — are context-dependent because they use many assumptions known or assumed to be true in the context of the specific reasoning. However, purely logical consequence seems to be context-independent, and it is sometimes operative in the binding mathematical as well as real-life conclusions.

Whereas "all" seems context-free, there are many quantifier expressions, and some are context-independent, while others are not. Similarly, sometimes the logical consequence is hidden inside an inference, while much more often we encounter strong inferences, good enough for practical purposes, but not logically valid. The two types of examples suggest the general problem, here applied to logic only, namely, how to characterise the context-free logical concepts in their natural environment, that is, in the field of their context-dependent associates. This approach is generally not adopted in logical considerations, even in the philosophy of logic. The focus usually is on the strictest notions, the ones most independent of context and the easiest to treat in a formal way. The other concepts, like the quantifier "many" or inference by abduction, are treated as strongly disconnected from the familiar logical formal(ised) concepts.

To consider the context-free notions as special, maybe extreme, cases in a broader field of related context-dependent notions can hopefully shed light on all these concepts and seems to agree more with the "man on the street" approach, for whom presumably all notions are context-sensitive. It would be good to have a general method or approach covering all such situations, but there is no guarantee that a uniform way of characterising context-independence is possible. Below, the two above-mentioned issues, that is, quantifiers and inference, are analysed separately, using distinct approaches. The issue of quantifiers is treated in the way presented with more details by Krajewski (2018). A general thesis is formulated: among all quantifiers, the context-free ones are just those definable by the universal quantifier. The issue of inferences is treated according to the approach of Epstein (1998; 2002 – see also his 2011–2015), where, however, the subject is not presented as a study of context-(in)dependence. To be sure, this treatment of arguments is not fundamentally novel; it does stress, however, better than more standard accounts, the unity of all inferences. The valid ones constitute an extreme case, the result of the disappearance of the need to look for counterexamples, or of the lack of context-dependence. This approach can be applied, as mentioned first by Krajewski (2012), to an analysis of a form of abduction called "reductive inference" in Polish literature on logic since Łukasiewicz (1911).

2. Context-free Quantifiers Among Quantifiers

In logic, from Aristotle to Frege to mid-20th-century predicate logic, only two quantifiers were incorporated: the general and the existential. They are still the only ones taught in basic general logic courses. Because in classical logic $\exists$ is the dual of $\forall$, that is, $\exists = \neg\forall\neg$, we can say that only the universal quantifier is added. Some other logics, such as intuitionistic logic, lack interdefinability, but non-classical logics are excluded from the present analysis.

From the perspective of linguistic realities, it is not clear why the general and existential quantifiers are the only concepts distinguished from among all possible quantifier expressions of the natural language. In natural language, there are dozens of quantifier expressions, that is, expressions that state or estimate the number of objects of a certain kind or the size of a collection, or compare sizes, etc. They include phrases like "all", "always", "nowhere", "almost never", "most", "infinitely many", "many", "from time to time", "a few", "quite a few", "several", "just one", "at least one", "an overwhelming part of", "as many as", "roughly as many as", and many more. In mathematics, some other quantifier expressions are used, for example "there are finitely many", "there are uncountably many", and "the set of … is dense in …", and the phrases like "almost all" are given various precise meanings in specific mathematical theories.

What could be the reason for the distinguished role of the familiar quantifiers in logic? First, simplicity. "All things" corresponds to the full set — either of all things or of all things in our universe of discourse. "At least one thing" corresponds to the notion of non-empty set. At the same time — and this is the second reason — we can see the general quantifier as an abstract counterpart of the operation of generalisation, one of our most important mental faculties.

The third reason for the distinguishing of $\forall$ and $\exists$ from among all possible quantifiers has to do with logical complexity. The number of nested quantifiers is a good indicator of logical complexity. The quantifiers $\forall$ and $\exists$ provide an excellent measure of complexity if the number of alternating nested quantifiers is counted. The realisation of this possibility gave rise to the Kleene–Mostowski hierarchy, classifying the sets obtained from recursive sets by a series of projections and complements. Other similar growing chains of ever more complicated objects have been established, such as the analytic hierarchy. From such a perspective, these simple familiar quantifiers look like anything but trivial. It is also of interest that neither Aristotle nor other pre-modern logicians considered nested quantifiers. The power of quantifiers, even the simplest ones, is seen only when several are combined. This is the fourth reason: these simple quantifiers bring much more expressive power than it would seem at first glance. When the standard additional machinery available in logic is employed, many new quantifiers can be defined. Some of them can be easily defined within first-order logic, for instance the numerical quantifiers: "there are exactly n", in short $\exists^{!n}$, "there are more than n", in short $\exists^{>n}$, and their combinations (like "there are three or four"), etc.

In higher-order logics and in set theory, many more quantifiers can be defined. Definitions in mathematics are expressed in the technical language of a given branch, but logicians have been able to express these definitions in the language of logic. Thus, for instance, "there are infinitely many" cannot be defined in first-order logic, but it can be defined in second-order logic. The Henkin quantifier, the first example of a branching quantifier, namely "for every x there exists y, and independently of that for every z there exists t such that $R(x, y, z, t)$", also goes beyond first-order logic, even though it reflects such a way of using the familiar quantifier expressions corresponding to $\forall$ and $\exists$ that can be found in natural language; this quantifier is easily defined in second-order logic: "there exist functions f, g such that for every x and for every z $R(x, f(x), z, g(z))$". The phrase "there are uncountably many" also defines a quantifier, but it makes sense only in reference to a background set theory. Unexpectedly, this quantifier can be recursively axiomatised (Keisler, 1970). There are many more examples of mathematical quantifiers. They suggested to mathematical logicians the concept of a "generalised quantifier".

Generalised quantifiers were introduced to logic by Andrzej Mostowski in 1957. The formula $(\mathrm{Q}x)\varphi(x)$ is satisfied in a model $\mathrm{M} = (M,\dots)$ iff the set $\{a\colon \mathrm{M} \models \varphi[a]\}$ belongs to the family of subsets of M that serves as the interpretation of Q. (Thus, $\forall$ is interpreted as $\{M\}$ and $\exists$ as the family of all non-empty subsets of M.) This notion was useful but was not sufficient for many formulations that are used in natural language. Mostowski quantifiers are all of type $\langle 1 \rangle$. A more general definition was introduced in 1966 by Per Lindström, who allowed quantifiers of an arbitrary type $\langle n_1,\dots,n_k \rangle$ that bind more variables and apply to several formulas, and in a model M are interpreted as relations between subsets of M (in the case of monadic quantifiers of type $\langle 1,1,\dots,1 \rangle$) or, more generally, relations between relations on M.

Mostowski and Lindström were mathematicians, so they made an important assumption which obviously seemed necessary to them: they considered only the quantifiers that are invariant with respect to isomorphism. The assumption in the case of monadic quantifiers amounts to the fact that only the size of the sets defined by the quantified formulas matters (cf. Peters & Westerståhl, 2006; Westerståhl, 2016). The assumption that logic should be completely topic-neutral constitutes the reason for admitting into logic only the quantifiers invariant under isomorphisms. Other mathematical properties can be defined by isomorphism-preserving quantifiers. Yet they are insufficient to model some quantifier expressions commonly used in natural language.

It is clear that logic is poorly equipped, if at all, to deal with many of the quantifier expressions listed above. For example, the concept "many" is different from the more logical quantifiers and seems hardly definable in general, since its meaning depends on the situation in which the term is used. It is seriously context-dependent. Peters and Westerståhl (2006, p. 213) call it "strongly" context-dependent, and some authors call it intensional. To evaluate a sentence with such a context-dependent quantifier, we need an appropriate understanding of the world, or at least of the appropriate fragment of the world. Logic itself is not sufficient. To know whether it is true that *many* women at my university are pregnant or that *many* have been to the Himalayas, we need to know how many women of a given age are, on average, pregnant, and how many go to the Himalayas. It is similar with quantifier expressions like “a few”, “several”, “a huge number”, “rarely”, “often”, etc., and even more obviously with “surprisingly many”, “almost everyone”, “virtually nowhere”, etc.

Despite the initial impression that the quantifier “many” is not definable, one could try to define it formally, or to model it, by adding a variable σ and defining “many” as more numerous than or equal to (the interpretation of) σ. This new variable can be either a numerical one, interpreted as a cardinal number, or a set variable, interpreted as a certain set S. Then “*many x*’s (satisfying φ)” is defined as having at least as many members as S, or as the requirement that the cardinality of the set of the values of x that satisfy the interpretation of φ is not smaller than the cardinality of S. The set S depends on the context; it is chosen specifically for each interpretation.

The problem with this attempt is that the definition of “a few” is the same, only with “<” instead of “>”. And the phrase “more than a few” is formalised exactly as is “quite a few” and “many”. And do we normally identify “many” with “more than a few”? Hardly. So, everything depends on the context, and introducing σ is of no help. Only the context counts.

An important feature of this formalisation is that “many” defined as “more than σ” is not invariant with respect to isomorphisms. What is more, the quantifier “many” lacks some monotonicity properties. It may happen that

$$\mathrm{M} \models [(\forall x)\ (\varphi(x) \rightarrow \psi(x))]$$

and still

$$\mathrm{M} \models [(Many\ x)\ \varphi(x)\ \&\ \neg(Many\ x)\ \psi(x))].$$

For example, if in my class of 20 persons at the University of Warsaw there were 8 students reading entire books, including each of the 7 who are pregnant, there would be many pregnant students and not many readers in the class.

It was just mentioned above that when linguistic quantifier expressions are reconstructed within logic, the requirement of context-independence is formulated as invariance with respect to isomorphisms. It seems that context-independence means that any extralogical terms referring to some specific fragments of the world are irrelevant for the understanding of the formula. The topic covered in the statement is of no consequence — only logic counts. The idea is that there is no need for any specific knowledge about the world, its physical or social aspects.

To attempt another thesis identifying the context-independent quantifiers, let us briefly recall what kind of thesis is meant here. Church’s thesis is the best-known example of a thesis identifying a formal concept with an intuitive one. The mathematical concept of recursive function is identified with the intuitive concept of effectively computable function. For a long time, the general conviction was that such a thesis can be justified by various arguments, but there is no way to prove its correctness because the intuitive concept is too vague to be part of a proof. However, in recent decades there have been various attempts to prove the identification. Namely, a proper analysis of the intuitive concept of computability can provide principles that make possible a demonstration that a function satisfying them must be recursive. There are more examples of similar theses, for instance “the Cantor–Dedekind thesis” that real numbers are defined by the appropriate set theoretic constructions. (For a discussion of Church’s thesis and the other examples as well as references to the literature, see, e.g., Krajewski, 2014.)

Now, it is the context-independence applied to quantifiers that is the intuitive notion we want to characterise. In addition to topic neutrality and invariance under isomorphisms, we can try to look at the ways the quantifier can be defined. It seems that whatever definition is formulated, it cannot be expressed without taking some specific logic into account. This is because quantifiers are logical objects that function within a logical framework. On the other hand, it would be necessary to emphasise their logical nature but ignore any specific logic. Any quantifier Q can give rise to a "logic" L(Q). Then Q is trivially definable in this logic. To avoid this triviality, let us call a (classical) logic *basic* if it is first-order, second-order, n-th order, type theory, or set theory. Hence the following thesis: A quantifier is context-independent iff it is definable in some basic logic. Because the common part of all such logics, as far as quantification is concerned, is the universal quantifier ∀, we can reformulate the thesis as follows: A quantifier is context-independent iff it is ∀-definable in some (basic) logic. Since we admit definability either in first-order or second-order or higher-order logic or in (formalised) set theory, and the general quantifier appears in each of these logics, we can say in short:

A quantifier is context-independent iff it is definable in terms of ∀,

or briefly,

Thesis on Quantifiers: Context-independence = definability in terms of ∀.

It is seen that the position of the general quantifier, or rather of our two familiar quantifiers, ∀ and ∃, is vindicated. This is the fifth — in addition to simplicity, the faculty of generalisation, the measuring of complexity, and the expressive power — and rather unexpected reason for distinguishing ∀: in the presence of the appropriate amount of logical machinery but with no generalised quantifiers, ∀ suffices to define all context-independent quantifiers. Thus, the power of the universal and existential quantifiers is claimed to be even stronger than it seemed on the basis of the definability of so many quantifiers by ∀, which is the empirical evidence for the quantifier thesis. According to this thesis, the power of ∀, at least in relation to quantifiers, extends to the whole realm of context-independence.

3. Context-free Inferences Among Inferences

Logicians like to say that a good inference is the one that a) is valid and b) whose premises are true. In everyday life, however, there are good inferences which have plausible but not certain premises, and the conclusion is made more plausible by the premises, but still might be false even when the premises are true. To present the two types of arguments in a uniform way, let us say that an inference is valid, resp. strong, if it is impossible, resp. hardly possible, for the premises to be true and the conclusion false at the same time. Thus, if an inference is (merely) strong, there is a possible way for the premises to be true and the conclusion false, but all such ways are unlikely, hopefully highly unlikely. This approach, proposed by Richard L. Epstein (1998; 2002), follows the accepted logical tradition, but remains particularly close to everyday arguments. The uniformity of the approach is methodologically and didactically satisfying.

Obviously, strong inferences can be more strong or less strong, whereas valid ones do not admit degrees, as they cannot be less than fully valid. It is also clear that valid arguments are just the extreme instances on the scale of the strength of arguments. What is more, the strength of strong arguments must be determined subjectively, because we need to imagine how probable it is to have a counterexample, that is, a situation in which the premises are true and the conclusion false. Our knowledge of the subject matter of the premises and the conclusion is essential. A strong argument that is not valid is so not due to the logical form only. Still, its strength can result from an application of a formal scheme, for example of the form: "Almost all S's are P; r is S; therefore, r is P", which is an excellent example of a strong but invalid inference. Does its form mean that it becomes a formal inference? Not really. The quantifier "almost all" is imprecise and, even more important for the topic of this paper, it is highly context-dependent. Its interpretation depends on the nature of the phenomena to which it is applied.

Context-dependent quantifiers bring problems that do not occur in context-independent logical considerations. It is possible to imagine schemes of seemingly very strong inferences, involving informal quantifiers, that are not valid because they fail in special circumstances, for example in the domain of infinite sets. Take, for instance, the inference "Almost all A's are B; many A's are C; therefore, some B's are C". In the realm of finite sets, this is a valid inference. Yet its premises seem to be true and the conclusion false if A is the set of all sets of natural numbers, B consists of all elements of a fixed ultrafilter on A (which is one of the standard mathematical interpretations of "almost all"), and C is the set of all finite sets of natural numbers.

For every general treatment of strong arguments, it is necessary to admit plausible but not necessarily certain premises. This assumption reflects our common mode of reasoning. Then some problems occur that are absent in inferences that are evaluated only on whether the premises and conclusion could be true or false. For example, when there are very many premises, the uncertainty can be compounded so that a false conclusion results. A good case is provided by the "lottery paradox": if all 10 million lottery tickets are sold to a similar number of people $p_1, p_2, \ldots, p_n$, we can safely assume that "p_1 doesn't win", "p_2 doesn't win", …, "p_n doesn't win" are all virtually true; after all, the probability that a given ticket does not win is 99.99999%. From this, a valid inference, by complete induction, can be made that nobody wins, which is patently false.

Strong inferences are of interest not only because they are close to everyday arguments. Another application of the concept of a strong inference can be made to the concept of explanation, that is, searching for reasons of observed phenomena. Often called abduction, sometimes induction (see Douven, 2017; Epstein, 2002), or in Polish logical literature "reduction" or "reductive inference" (see Łukasiewicz, 1911), it is usually considered as a means of reasoning not only different from deduction, but sharing no common ground with it. Yet it can be seen as another example of a strong inference. One way to describe it is as follows. We observe (the truth of) B and want to find its reason, that is to say, an A such that $A \Rightarrow B$, where the symbol "$\Rightarrow$" denotes the reason, which is much more than the material implication $A \rightarrow B$. What it means is a difficult problem, as the logical analysis of the concept of explanation has no standard formulation (see, e.g., Epstein, 2002). Anyway, what we do, either explicitly or intuitively, can be rendered as an analysis of possible causes of B and rejection of all of them but one, that is, A:

$$(*) \qquad B,\ A \Rightarrow B,\ [A_1 \Rightarrow B, \neg A_1],\ [A_2 \Rightarrow B, \neg A_2],\ [A_3 \Rightarrow B, \neg A_3],\ \ldots\ /\ A,$$

where "/" indicates inference.

To get a formulation closer to the usual logical calculus, we can model the causal relation $A \Rightarrow B$ by $(\forall s)\ (A^{(s)} \rightarrow B^{(s)})$, where s corresponds to a situation. That is, for each situation s, if A related to or applied in that situation holds, then B also holds in that situation. Similarly, for $A_i \Rightarrow B$, we have $(\forall s)\ (A_i^{(s)} \rightarrow B^{(s)})$. Finally, we replace A, B, $\neg A_i$ by $A^{(c)}$, $B^{(c)}$, $\neg A_i^{(c)}$, respectively, where c is the current situation that is being considered at the moment in which we want to find an explanation. Then the inference is of the form:

$$B^{(c)},\ A^{(c)} \rightarrow B^{(c)},\ [A_1^{(c)} \rightarrow B^{(c)}, \neg A_1^{(c)}],\ [A_2^{(c)} \rightarrow B^{(c)}, \neg A_2^{(c)}],\ \ldots\ /\ A^{(c)},$$

which is a simplified form of the original (*). This notation is simpler and possibly more suggestive, but it is far from clear whether every causal relation can be reduced to a general statement of the kind $(\forall s)\ (A^{(s)} \rightarrow B^{(s)})$. Some authors do claim that explanation should involve reference to a general law (cf. Epstein, 2002, p. 249).

Whatever analysis of explanation and notation is followed, the key for the assessment of the inference is how well we can imagine and reject the possible reasons A_i's. We need to know the subject matter of B, the circumstances in which it can appear, its possible causes, etc. The better we know them, the more adequate is the list of possible additional premises of the form $[A_i \Rightarrow B, \neg A_i]$. The longer and more complete the list is, the stronger is the conclusion, because it is less and less probable for all the premises to be true and A false. Usually, there is no way to make the list complete; various wild possibilities are imaginable, even if hardly possible. We can only hope that we can realise all realistic possibilities. The inference is then strong even though it remains fallible: it is still possible, though improbable, for all the premises to be true and the conclusion false.

Let us assume that the event B must have a cause, which seems to be an instance of the Leibnizian principle of sufficient reason that lies at the foundations of the scientific worldview. Then we can say that the following is true:

$$B \rightarrow (A \vee A_1 \vee A_2 \vee A_3 \vee \ldots).$$

This premise is implicit in (*). What we get is:

$$B,\ A \Rightarrow B,\ [A_1 \Rightarrow B, \neg A_1],\ [A_2 \Rightarrow B, \neg A_2],\ [A_3 \Rightarrow B, \neg A_3],\ \ldots$$
$$B \rightarrow (A \vee A_1 \vee A_2 \vee A_3 \vee \ldots)$$
$$/\ A.$$

Now, it is evident that in the above inference the formulas with "$\Rightarrow$" are not really needed; they indicate that the formulas A_i are not arbitrary sentences but *bone fide* causes. (As noticed by a referee, the expression "$A \vee A_1 \vee A_2 \vee A_3 \vee \ldots$" suggests an existential quantifier, "there exists a cause". The problem of inference is thereby related to the problem of quantifiers.) The impact of the new premise is especially clear if we are able to list *all* possible causes of B. Then we can claim that

$$B \rightarrow (A \vee A_1 \vee A_2 \ldots \vee A_n),$$

and have a valid inference, not just a strong one, using this added premise. In this case, the premises $A_i \Rightarrow B$ are completely superfluous. Removing them, we get a valid deductive inference:

$$B,\ B \rightarrow (A \vee A_1 \vee A_2 \vee \ldots \vee A_n),\ \neg A_1,\ \neg A_2,\ \ldots,\ \neg A_n\ /\ A.$$

If there is another possible cause A_0 such that $A_0 \Rightarrow B$ and there is no reason to claim that $\neg A_0$, the conclusion must be modified to $A \vee A_0$.

In order to connect the issue of strong inferences to context-(in)dependence, let us repeat that valid arguments are extreme cases of strong ones. Strong arguments are heavily dependent on the context. We need to use our knowledge of the situation to which the argument refers and imagine possible ways in which the premises can be true and the conclusion false. Logicians love the fact that valid arguments can be infallible due solely to the logical form of the premises and the conclusion. Then inference is made by an application of a law of logic, or a tautology in a logical calculus, and the dependence on context disappears. Only the form counts. Thus, among the strong arguments, the deductive ones, that is, those based on logical tautologies (in some logical system) can be seen as the context-free ones because only their form counts and their contents are irrelevant. Generalising from deductive to all valid arguments, including those that might not be expressible as a tautology in some logical calculus, we get the tentative

Thesis on Inferences: A strong inference is valid iff it is context-independent.

While the implication "validity $\rightarrow$ context-independence" is formed by generalisation, it can be justified by a belief that validity always results from deductiveness, which is another strong thesis. The reverse implication, that context-independence of a strong inference forces its validity, is based on the idea that non-validity would require a possible counterexample (true premises and false conclusion) and that would indicate the context on which the inference is dependent.

4. Conclusion

Two fundamental logical notions — quantifier and inference — have been analysed from a hitherto unrealised angle. In each of the two cases, strict variants of the notion have been identified as context-free varieties of a more general category in which many less strict but useful and widely used variants of the notion are present. It would be of interest to find out whether it makes sense to analyse other concepts in a similar way.

The kind of context-dependence to which our present analysis referred was not the same for the two notions: in the case of quantifiers, the meaning depends on the situation in which a quantifier expression is employed; in the case of inferences, the relation between premises and conclusion depends on possible situations in which the premises would be true and the probability of their occurrence. For quantifiers, the *Thesis on Quantifiers* has been proposed, identifying the context-independent ones with those definable in logic by the universal quantifier, the prime example of a context-independent quantifier. The tentative *Thesis on Inferences* identifying the validity of a strong inference with context-independence seems reasonable, but to justify it, one should exclude the possibility that some inferences can be valid for a good reason other than being deductive, that is, based on tautology in some logical system, so that they could be context-dependent in some way.

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