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Michał Eckstein

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What is a particle?

Michał Eckstein*†

†Institute of Theoretical Physics, Jagiellonian University, Łojasiewicza 11, 30-348 Kraków, Poland
‡International Centre for Theory of Quantum Technologies, University of Gdańsk, Wita Stwosza 63, 80-308 Gdańsk, Poland
*Corresponding author. Email: michal.eckstein@uj.edu.pl

Abstract
This is a short reflection on the notion of a 'particle' in particular and on the methodology of physics in general.

Keywords: particle, philosophy in science, quantum field theory

1. Introduction

According to the (*Oxford English Dictionary*) a *particle* is: "A minute fragment or quantity of matter; the smallest perceptible or discernible part of an aggregation or mass.". This agrees with the common-sense definition of particles as 'small things, out of which the bigger things are made of'.

Clearly, such a concept of particle is deeply rooted in the atomistic vision of the physical world, which — in turn — is supported by the development of physics in the past two centuries.

In philosophy of physics the 'particle' is often treated as a *primitive notion* (cf. (Eckstein and Heller 2022)), that is a concept which is "immediately understandable" and "employed without explaining its meaning" (Tarski 1994). The simplest example of such a primitive notion is that of a *point* in geometry.

Within an axiomatic approach, the primitive notions are utilised to spell out the axioms. Consequently, they are taken for granted, though they might be connected or restricted by the axioms. For instance, in geometry the axiom: "For every two points there exists a line that contains them both." links the primitive notions of a point and a line.

In this spirit, particles as primitive concepts appear, in parallel to "light rays" in the celebrated Ehlers–Pirani–Schild (EPS) axiomatisation of relativistic spacetime (Elhers, Pirani, and Schild 1972). More precisely, the authors take 'particle' to mean a "worldline of a freely falling particle". In either case, on the notions of particles and light rays a quasi-operational axiomatic system of a Lorentzian spacetime is established. As the authors admit, the particles are understood in the classical sense as "bodies whose extension and structure can, under suitable circumstances, be neglected". In fact, assuming that the particles are quantum or, more generally, non-classical, may lead to very different structures and axiomatics (Adlam, Linnemann, and Read 2022; Eckstein and Heller 2022).

2. Particles in classical and quantum physics

2.1 Classical theory

The notion of a particle adopted in the EPS axiomatics, which is in fact the intuitive one, stems from classical (*i.e.* 'before-quantum') physics. A *classical particle* is a pointlike object with certain inherent properties, such as mass, charge, position, momentum etc. Some of these properties are invariant, e.g. mass or charge, and some can change, e.g. position or energy, through free evolution or interactions with the environment. A single classical particle is treated as a passive 'test' object, the properties of which are influenced by the background forces (gravitational, electromagnetic, ...), but the background is *not* influenced by the particle. In other words, it is typically assumed that the backreaction of the particle on its environment is negligible, unless we consider a large number of particles, e.g. constituting a dust cloud.

Such understood classical particle is treated merely as an idealisation of *real* particles. On the fundamental level, the particles are assumed (still within classical physics) to act on whatever environment they are embedded in. This stems from a deep, though seldom phrased explicitly, methodological principle that *physical interactions are always mutual*.

The main point here is that the employment of the notion of a classical particle involves an ontological committent. While the formal concept is an unphysical idealisation, we do assume that there exist in Nature 'real particles', which are adequately modelled by classical particles. In other words, we assume that in a physical experiment we study the actual real particles and describe them formally as classical particles. Such a viewpoint is coherent with the common-sense intuition on empirical sciences — we observe real phenomena and attempt to model them mathematically with idealised concepts.

2.2 Nonrelativistic quantum theory

The quantum theory, however, drives us away from the common-sense viewpoint on particles. In non-relativistic quantum mechanics a 'particle' is still a primitive notion. Concretely, a *quantum particle* is a quantum system (the 'system' is itself a primitive notion) described by a noncommutative algebra of observables generated by position ($\hat{x}$), momentum ($\hat{p}$), and possibly some internal degrees of freedom (Strocchi 2008). It may have some objective (i.e. classical) properties, such as charge or mass, but it does not have a definite position nor momentum. The latter properties actualise (randomly!) only upon an active measurement[1].

A quantum particle is also an idealisation of a 'real' particle, but in a somewhat more convoluted sense. In the quantum world, a real particle is never completely isolated from its environment. Its (quantum) degrees of freedom get *entangled* with the degrees of freedom of the environment, hence, at the fundamental level, one must treat the particle and its environment (which is, actually, the entire physical Universe) as a single global system.

A quantum system is completely characterised by a *state*, that is a density operator on a Hilbert space $\mathcal{H}_P$. An isolated quantum particle is characterised by a pure state, that is a vector $|\psi\rangle \in \mathcal{H}_P$. This state is *intrinsic* — it pertains only to the particle itself and not to any external systems, such as 'environment', 'detector' or 'observer'. It could thus be seen as a quantum analogue of particle's properties. But whenever a particle interacts with its environment it becomes entangled with it, so that the total (pure) state of the system is a superposition of different states of the particle and the environment, $|\Psi\rangle_{P+E} = \sum\!\!\!\!\!\!\int_i c_i |\psi_i\rangle_P |\phi_i\rangle_E \in \mathcal{H}_P \otimes \mathcal{H}_E$. Furthermore, this decomposition is not unique — as the values of the coefficients c_i depend on the choice of the bases $\{|\psi_i\rangle\}$ and $\{|\phi_i\rangle\}$. Consequently, it is no longer possible to assign a unique vector in $\mathcal{H}_P$ to the particle itself. We thus see that, upon interaction with the environment, a quantum particle not only changes its state (i.e. 'quantum properties'), as it is the case for a classical particle, but actually it *looses its identity*.

At the operational level, a quantum particle is completely characterised by the algebra of observables that correspond to all possible measurements, which can be performed on it (Strocchi 2008). The possible outcomes of these measurements, along with the respective probabilities of occurrence are encoded in the particle's reduced density operator $\rho_P = \mathrm{Tr}_E |\Psi\rangle_{P+E} \langle\Psi| = \sum\!\!\!\!\!\!\int_i |c_i|^2 |\psi_i\rangle_P \langle\psi_i|$. Note

1. One may attempt to save the common-sense viewpoint, for instance by adopting the Bohmian interpretation of quantum mechanics. Then, the particles do have intrinsic properties, in particular, trajectories, but the price to pay is the introduction of a fundamentally unobservable object – the 'pilot wave'. A serious drawback of Bohmian mechanics, which undermines its philosophical implications, is that it is not compatible with the theory of relativity (see, however, (Dürr et al. 2014)).

that while ρ_P is does not depend on the choice of the Hilbert space basis, its decomposition using $|\psi_i\rangle_P$ does. But the state ρ_P does not provide a complete description of the quantum particle, because it does not contain the information about the complex phases of the coefficients c_i. In a single measurement of a given observable A, with a spectral decomposition $A = \sum_a a\mathcal{P}_a$, we only register a single outcome a with the probability $\mathrm{Tr}\,\mathcal{P}_a\,\rho_P$. However, given a large collection of particles with the (effective) state ρ_P, or using weak measurements (Dressel et al. 2014), one can perform a *quantum state tomography* and reconstruct the state ρ_P to an arbitrary precision (Paris and Rehacek 2004; Wu 2013). One could thus say that the state ρ_P is a certain idealisation of a real quantum particle, which we can access empirically.

Finally, let us observe that a classical particle can also be seen as an idealisation of a quantum particle. Indeed, if the scales of the experimental setup are macroscopic, as it is surely the case for instance in astronomy, then one can safely assign an average classical trajectory to a particle, $x(t) = \mathrm{Tr}\,\rho_P(t)\hat{x} \approx \langle\psi(t)|\,\hat{x}\,|\psi(t)\rangle$ and we can neglect the particle's entanglement with its environment. In other words, if the resolution of the measuring device is much larger than the width of the quantum wave packet, then the particle at hand is effective classical.

2.3 Quantum field theory

The concept of a particle becomes even more cumbersome within the relativistic quantum. In fact, in quantum field theory a particle is no longer a primitive notion. It appears as a specific state of a quantum field — a *single particle state* —, which is a certain vector in the Fock space. The point is, however, that in such states exist only in a free, i.e. non-interacting, theory. For general interacting quantum fields states with a fixed number of particles appear only asymptotically — as "in" and "out" states for the S-matrix (Haag 1996).

Even at the operational level we cannot unequivocally identify a relativistic quantum particle. This is because any single-particle detector (and in fact any QFT detector at all) has a non-vanishing vacuum response (Reeh and Schlieder 1961; Peres and Terno 2004). In other words, a detector click may be induced by random fluctuation of the QFT vacuum[2].

We thus see that the realm of relativistic quantum theory is very different from the common-sense one: At the fundamental level there are no particles, only quantum fields[3]. In specific circumstances, for instance in detection events, one can (statistically) identify 'single-particle' phenomena. Furthermore, the properties of QFT particles are even more cumbersome than these of non-relativistic quantum particles. Firstly, a single-particle state has a non-vanishing probability of detection in any region of spacetime (Reeh and Schlieder 1961; Peres and Terno 2004). Secondly, the mass of a particle is a concept which depends on the energy scale[4]. On the other hand, the charge of an elementary QFT particle is a *classical* property, as the superselection rules forbid the existence of quantum superpositions of states with different charges.

3. An information-theoretic perspective

A rather different concept of a particle emerges from modern information theory, inspired by the device-independent approach to quantum information (Pironio, Scarani, and Vidick 2016). It focuses on the information-processing aspects of phenomena, while neglecting the physical details of the involved objects. In this context a 'particle' is a primitive notion and signifies merely a physical '*information carrier*' (Brunner et al. 2014; Eckstein et al. 2020; Miller et al. 2021). This notion hinges upon another methodological principle that *information is physical*. It means that any information

2. This phenomenon has led to some confusion concerning relativistic causality and signal propagation in quantum field theory (Hegerfeldt 1994; Buchholz and Yngvason 1994; Sabin et al. 2011).

3. It is somewhat ironic that the Standard Model of Particle Physics is a theory, which says that in fact there are no particles in Nature.

4. The effect of 'running masses' is a consequence of renormalization procedure (Weinberg 1995).

(classical, quantum or else) must be supported in *some* physical system. But the information itself is independent on the physical system, in which it is encoded. Indeed, in any actual communication protocol the information is faithfully transferred with the help of a number of different physical systems: antennas, cables etc.

Such an information-theoretic particle is a concept, which is independent of the physical theory. When describing an information processing protocol we are only concerned with the data (bit, qubit, etc.) and not with the physical carriers. The latter can be classical particles, quantum particles, field excitations or some other, possibly unknown, physical entities.

Nevertheless, information theory is not purely epistemic and it does involve an ontological commitment. Indeed, we must assume that the information carriers are available for the agents planning to execute a given protocol. In other words, when considering admissible information processing protocols we do assume that suitable resources exist in Nature. Furthermore, these resources exist independently of whether the agents decide to use them or not. Bringing the reasoning back to fundamental physics, we can say that quantum fields exist in Nature and free agents can use them (i.e. interact with them in a designed way) to process information.

The gist of the modern information-theoretic approach is that it is based on operational notions only. A particle in the information-theoretic sense in indeed operational, because it assumes that there must exist a physical device and an observer able to access the information carried by it. This might suggest a methodological guideline for the formulation of physical theories — they ought to be based solely on operational notions.

This guideline is often adopted in the context of quantum foundations because of the success of 'operationalising' the axioms of (non-relativistic!) quantum mechanics on the one hand (see (Chiribella and Spekkens 2016) and refs therein) and performing 'theory-independent' experiments on the other. The latter are variants of the famous EPR–Bell experiment, which indeed can be formulated purely operationally, without invoking the notions from any particular physical theory (see e.g. (Brunner et al. 2014)):

> Take two measuring devices A and B, each with two possible measurement settings ($x, y = 0, 1$) and two possible outcomes ($a, b = \pm$), and feed them with two particles coming from a common source. Arrange the setup so that the detection events at A and B are spacelike separated, the devices register at least 83% of particles and the measurement settings are random[5]. Gather a large detection statistics from many particles and different measurement settings and compare the correlations for a given pair of settings:
>
> $$\begin{aligned} C(x,y) &= P(a=b\,|\,x,y) - P(a=-b\,|\,x,y), && \text{for all} \quad x,y \in \{0,1\} \\ S &= C(x,y) + C(x',y) + C(x,y') - C(x',y'), && \text{for all} \quad x,x',y,y \in \{0,1\}. \end{aligned}$$
>
> If for some settings $S > 2$, then the experiment cannot be explained by *any* local hidden variable theory and if $S > 2\sqrt{2}$, then it cannot be explained by the quantum theory (relativistic or not).

We see that such an experiment provides a very powerful tool to test our mathematical theories against the empirical data. Its implementation requires no knowledge of any physical theory[6] and no a priori rules on how to model the 'particles' and 'detectors'.

But all that concerns an EPR–Bell experiment *in principle*. The actual EPR–Bell experiments are performed with sophisticated setups, the design of which is heavily based on the established physical theories, including quantum mechanics. What is more, physicists knew from the very beginning

5. These are the famous "loopholes" in Bell-type experiments. The two former can be, and have been, closed (Aspect 2015), while the third one is in fact a methodological principle, and hence it cannot be closed (Eckstein and Horodecki 2020).

6. Well, almost... We do need to know at least what "spacelike separated" means.

that in order to test quantum mechanics against local hidden variables one needs to engineer a specific quantum state, which is (as close as technically possible to) a maximally entangled state. In conclusion, while the EPR–Bell test, as described above, is formulated in a model-independent manner, its implementation must be done with concrete a physical system, of which we have a good understanding and control. But this of course requires a reliable and well-established theory describing the physical system at hand.

4. Conclusions and reflections

It is certainly instructive and valuable to formulate the descriptions of physical experiments in terms of purely operational notions — as it was done for the Bell test. However, one has to keep in mind that in actual experiments the physicists are bound to use complex theoretical schemes. The latter are based on a number of (typically not primitive) notions, which *only* make sense within these schemes. Consequently, the published results of experiments are hardly even understandable to non-experts.

Take, for instance, the following excerpt from the Higgs boson discovery paper (Aad et al. 2012), worth a Nobel prize:

> *Clear evidence for the production of a neutral boson with a measured mass of* 126 ± 0.4*(stat)* $\pm$ 0.4*(sys) GeV is presented. This observation, which has a significance of 5.9 standard deviations, corresponding to a background fluctuation probability of* $1.7 \cdot 10^{-9}$*, is compatible with the production and decay of the Standard Model Higgs boson.*

The "Higgs particle" is in fact a new quantum field with specific properties (charge, mass, spin) and coupling to other known quantum fields. The reported evidence for its existence came from the detection of its decay products of the type and rate "compatible with the Standard Model".

We thus see that the notion of a particle stemming from modern physics is rather far from the common-sense intuitive notion of a 'fundamental portion of matter'. What does it say about the fundamental level of the physical world?

If we take the Standard Model of Particle Physics at the face value we are pushed into the conclusion that what really exists are the quantum fields — ephemeral entities, which extend over the entire Universe. We have no direct access to them, even locally. In suitable experimental circumstances we can measure and identify the basic excitations of these fields, which we call the *elementary particles*. From these particles (that is, really, from quantum fields' excitations) atoms and molecules are made. This leads to a rather unsettling conclusion that all matter — trees, notebooks, stars and planets, this article and, indeed, ourselves — are eventually but quantum fields. On top of that, we face the notorious *measurement problem*[7]: We are soaringly missing an explanation on *why* and *how* we perceive a classical world, furnished with palpable objects, given that they do not exist at the fundamental level.

Despite the unquestionable success of quantum field theory, it seems unlikely that we have now reached the fundamental level of physics that would ever be accessible to humanity, although some of the most prominent physicists expressed such a viewpoint (Hawking 1988; Weinberg 1994). One can advance different arguments against such claims, for instance:

1. General Relativity is incompatible with quantum field theory (at least in the perturbative regime), so physics requires a unified 'quantum gravity' theory, which will surely change our views on the fundamental level of physics.
2. Any physical theory is based on mathematics and mathematics itself cannot be both complete and consistent — cf. (Hawking 2002)).

7. See Chapter 11 in (Landsman 2017) for a nice interdisciplinary take on this problem.

3. Any physical experiment involves an 'intervention' of an observer, which affects the studied system. Furthermore, the source of this intervention (that is the 'observer') is not modelled within the theory, which the experiment was designed to test. Consequently, physics will never be complete because of the irremovable tension between the existence of universal laws of physics and the demand of their testability (Eckstein and Horodecki 2020).

In order to "save the phenomena" philosophers try to build sophisticated 'interpretations' of the quantum theory. It appears to me that these efforts (which seem completely hopeless if one attempts to 'interpret' in this way the full Standard Model of Particle Physics) miss the point. And the point is that modern physics is irreducibly tangled with abstract mathematics. The debate on the ontic status of mathematical structures persists in the philosophical discourse since at least 2500 years. Now we have strong reasons to claim that this philosophical problem expanded into physics — see (Heller 2021). The result is that such seemingly primitive and intuitive notions as a particle are in fact a highly mathematicised concept. The title question: "What is a particle?" cannot be answered on purely empirical grounds. The answer must take into account two aspects:

(1) What is a particle in a mathematicised theory T?
(2) Is theory T consistent with empirical data?

Note, however, that even if the answer to the second question is positive, it does not imply that 'real' particles are indeed particles in the sense of theory T, for there can be many inequivalent theories, $T_1, T_2, T_3, \ldots$, all consistent with the available theoretical data. For instance, the quantum field theory, and string theory, and loop quantum gravity, and many other, are all consistent with empirical data, while they entail radically different concepts of a particle.

This brings us to the question of how do we determine, which physical theory is actually adequate to model natural phenomena. A good account of how it works in practice was given by Thomas Kuhn in his famous book "The Structure of Scientific Revolutions" (Kuhn 2012): The development of Science is a cycle with two periods — stable and revolutionary. During the first one, there is a ruling paradigm of an accepted theory T_1. A revolution can be triggered if there is a critical amount of experimental data, which are not explained by T_1, but can be explained by a new theory T_2. In consequence of a scientific revolution is a paradigm shift from T_1 to T_2. From this viewpoint, we are now in a stable period with the paradigm of quantum field theory. String theory, loop quantum gravity, and other quantum gravity schemes, aim at starting a revolution, but so far without success.

It should be stressed, however, that the structure of scientific revolutions is not purely sociological, but reflects a deep property of the natural world (Heller 2009). It often happens that a new theory T_2 is proposed first, although there is no urgent need for it. Its rigorous mathematical structure married with operational concepts fosters predictions of some new, unforeseen, phenomenon, which is eventually observed in a dedicated experiment. A good example is Einstein's General Relativity, which seemed to be an overkill to explain the tiny anomalous precession of Mercury's perihelion. But, as it happened, GR turned out to have a huge predictive power. We had to wait an entire century to be able to register gravitational waves, predicted by Einstein in 1916. And we could not possibly get the idea that such a phenomenon might exist if we stayed within the Newtonian paradigm.

It is mathematics, which allows us to go beyond the "known unknowns" towards the "unknown unknowns". In other words, a new theory allows us to formulate new operational questions, which could not have been formulated in the old theory. We often first need a new theory to propose a new experiment, which would put in question the old theory.

Does it mean that the old theory is falsified after a paradigm shift? Not at all! It simply means that we have reached the limits of its applicability. The new theory must reduce to the old one in a suitable formal limit (instance $\hbar \to 0$ or $c \to \infty$). But this does not mean that the new theory is better at explaining *all* experiments — the new ones and the old ones. Often, the new theory is quite useless for the description of phenomena well covered by the old theory. Indeed, it would not make much sense to use neither relativity nor quantum mechanics (let alone the Standard Model …) to explain the chaotic motion of a billiard ball — classical mechanics is much handier. In more philosophical terms, insisting on the reductionist picture leads us to the rather clumsy — and pretty useless — conclusion that the billiard ball is a very complicated bundle of quantum fields.

This suggests that scientific revolutions are in fact rather peaceful. The old theory is not guillotined from the physical world by the new one coming to power. It is simply put to a bastille, which limits its applicability, but does not prevent it from developing. Indeed, there was (and still is!) substantial development in classical mechanics, in particular in the domain of deterministic chaos, long after quantum mechanics and relativity became paradigmatic.

Consequently, the landscape of modern physics appears to be much more complex than the simplistic reductionistic picture — see Fig. 1. The latter implies an 'onion-like' structure of physical theories. A better analogy seems to be that of a manifold: Physical theories form an 'atlas' locally mapping different aspects of the physical world. For instance, quantum mechanics is the valid framework, roughly, when the considered length scales range from 10^{-13}m to 10^{-7}m. At much shorter distances, one crosses the energetic threshold for pair creation and hence quantum field theory is needed. At the other end, when objects become much larger in size than micrometers, then the quantum effects are negligible and classical theory provides a better description. The actual boarders between theories are usually misty and very interesting. In particular, the quantum-to-classical transition is being intensively studied, both theoretically and experimentally (Bassi et al. 2013; Carlesso et al. 2022). It is possible that it would bring a new theory, which does not belong neither to the classical nor to the quantum paradigm.

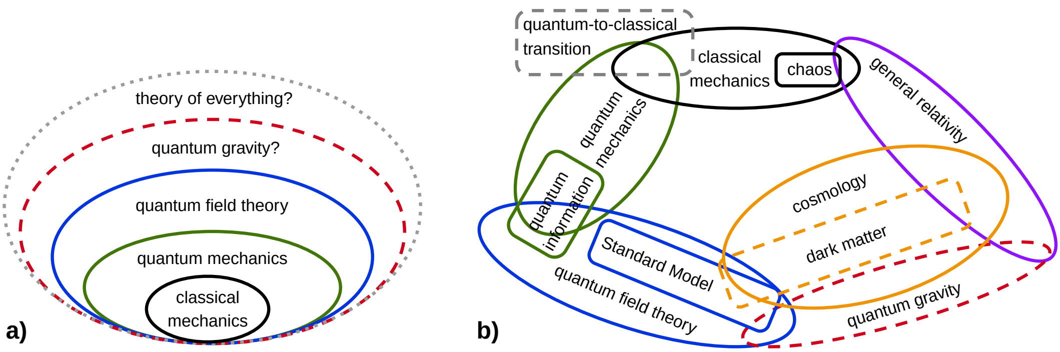

Figure 1. a) In the reductionistic vision physical theories form an 'onion-like' structure, in which every new theory generalises the old one, encompassing models of a larger class of natural phenomena. **b)** A better account of the actual status of physics is given by a 'manifold-like' structure, in which different physical theories agree (formally) at a certain overlap, but otherwise cover a different class of natural phenomena. The dashed lines signify domains, in which an empirically satisfactory theory is still missing.

Note that in the manifold picture the mythical 'quantum gravity' is reduced to a specialised theory encompassing a specific class of phenomena at the Planck scale. It might induce a revolution in cosmology (and, possibly, in particle physics), but it is not an all-embracing unified framework. Indeed, there will not be any use of quantum gravity, for instance, in deterministic chaos or atomic optics. On the other hand, these fields can and will develop independently of the status of quantum gravity. In the 'manifold' vision, there is no all-embracing 'theory of everything' (at least not in the usual sense of (Hawking 1988; Weinberg 1995)), but there are numerous uncharted regions marking the directions of possible future physical theories.

In this short note we have argued that the intuitive notion of a 'particle' is valid at macroscopic scales, but becomes more and more ephemeral when we ponder it at shorter scales and/or higher energies. We posit that this is actually the fate of *all* physical notions. There are no definite physical objects or phenomena, which can be named and described. They only exist within, and in the sense, of a specific theoretical framework, which has inevitable limits of applicability. The physical world must be contemplated from a holistic perspective, which takes into account a multi-layered irreducible structure of physical reality. And the latter is inevitably connected with mathematical structures modelling the different layers.

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2. Invitation to Functorial Spaces

Wiesław Sasin and Michael Heller

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Invitation to Functorial Spaces

W. Sasin*† and M. Heller‡

†Institute of Mathematics and Cryptology, Military University of Technology, Kaliskiego 2, 00-908 Warsaw, Poland
‡Copernicus Center for Interdisciplinary Studies, Jagiellonian University, Cracow, Poland
*Corresponding author. Email: first.author@address.edu

Abstract

We propose a generalization of the differential space concept that we call functorial differential space. It consists in replacing the standard differential structure $C^\infty(M)$ of a differential space by the differential structure consisting of functions from an extended space $\bar{M}(A)$ to an algebra A, satisfying certain general conditions, but not necessarily even commutative, and $\bar{M}(A)$ being a set of mappings $C^\infty(M) \to A$, interprtetd as new points. In this way, we obtain a whole family of theories of differential spaces (depending on the algebra A). It is this family that is a functorial differential space. An important class of spaces arises if we tensor multiply the differential structure of a functorial differential space by a Grassmann algebra. This leads to the concept of a functorial differential superspace (or supermanifold). In this context we also consider Einstein-Grassmann functorial super algebras.

Keywords: differential structure, functorial geometry, supermanifolds, Einstein-Grassmann algebra

1. Introduction

In papers (Pysiak et al. 2020) and (Pysiak et al. 2023), we used a method, called by us the functorial method, to study the infinitesimal structure of space-time and Einstein algebras. Broadly speaking, it consists in replacing, in the theory of differential spaces, the differential structure, consisting of smooth functions from a set M of points to the space of real numbers $\mathbb{R}$, i.e. functions of $C^\infty(M)$, with a structure consisting of smooth functions from an extended space $\bar{M}(A)$ to an algebra A; the latter is supposed to satisfy certain general conditions (smoothness condition could be relaxed); $\bar{M}(A)$ is here a set of mappings $C^\infty(M) \to A$ interpreted as a set of new points. The formalism is constructed in such a way that the algebra A plays the role of a stage from category theory, and the whole becomes a functor from the category of algebras of a certain type to the category of sets. Substituting various algebras for A, we obtain a family of theories of differential spaces. An important class of spaces arises if we tensor multiply the differential structure of a functorial differential space by a Grassmann algebra. This leads to the functorial superspace concept.

Regardless of its applications, the method of functorial differential spaces seems interesting in itself. In this paper, we will make a more systematic review of this method, as it has been developed so far, and refine some of its details.

J. A. Navarro Gonsàlaz and G. J. B. de Silas (Navarro Gonçàlez and Sancho de Salas 2003), and Jet Nestruev (Nestruev 2002) should be considered the precursors and at least partly the inspirers of our approach to functorial spaces. In (Navarro Gonçàlez and Sancho de Salas 2003), the authors introduced the concept of parametric points as mappings from one differential space to another; in (Nestruev 2002) the author considers the real spectrum of a certain geometric algebra A (that is, an algebra that can be represented as an algebra of real functions on a certain space) as a set of maps from A to $\mathbb{R}$. We should also mention R. S. Palais (Palais 1981), A. Kock (Kock 2006, 2009), and I.

Moerdijk and G. E. Reyes (Moerdijk and Reyes 2010) from whom we have borrowed and developed some ideas and methods.

The plan of our work runs as follows:

- In Section 2, we define the main environment of our analyses, i.e., C^∞-algebras.
- In Section 3, we consider a differential space $(M, C^\infty(M))$ and make a key move by defining a space whose points are maps from from M to an algebra A, and a differential structure on this space. The result of this procedure is a functorial ringed space whose geometric properties we investigate.
- In Section 4, we deal with the issue of generating differential structures for functorial differential spaces.
- In Section 5, we further develop the differential geometry for functorial spaces.
- In Section 6, we define functorial Einstein algebras.
- In Section 7, we construct sheaf version of functorial differential spaces.
- In Section 8, we show that if the differential structure of a functorial differential space is tensor multiplied by a Grassmann algebra, one obtains a functorial super differential space. Correspondingly, one can also speak of Einstein-Grassmann superalgebras.

2. C^∞-algebras

In this section, we recall the definition and some properties of the C^∞-algebra, which sets the context for our further constructions.

Definition 1 *A unital commutative $\mathbb{R}$-algebra A is a C^∞-algebra if, for any $n \in \mathbb{N}$, $\omega \in C^\infty(\mathbb{R}^n)$ and $a_1, \ldots, a_n \in A$, the element $\omega(a_1, \ldots, a_n)$ is defined and the following conditions are satisfied*

1. *for $\varphi, \psi \in C^\infty(\mathbb{R}^2)$, $a_1 + a_2 = \varphi(a_1, a_2)$, where $\varphi(x_1, x_2) = x_1 + x_2$.*
   $a_1 \cdot a_2 = \psi(a_1, a_2)$, where $\psi(x_1, x_2) = x_1 \cdot x_2$.
2. *for $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$, $\pi_i(x_1, \cdots, x_n) = x_i$, $i = 1, 2, \cdots, n, a_1, \cdots, a_n \in A$,*
$$\pi_i(a_1, \cdots, a_n) = a_i,$$
3. *for the function $1 \in C^\infty(\mathbb{R}^n)$, $1(x_1, \cdots, x_n) = 1$, $a_1, \cdots, a_n \in A$,*
$$1(a_1, \cdots, a_n) = 1_A,$$
4. *for $\theta \in C^\infty(\mathbb{R}^m)$, $\omega_1, \ldots, \omega_m \in C^\infty(\mathbb{R}^m)$, $a_1, \ldots, a_n \in A$, $m, n \in \mathbb{N}$,*
$$(\theta \circ (\omega_1, \ldots, \omega_m))(a_1, \ldots, a_n) = \theta(\omega_1(a_1, \ldots, a_n), \ldots, \omega_m(a_1, \ldots, a_n)).$$

Let A, B be C^∞-algebras. A homomorphism $f : A \to B$ of $\mathbb{R}$-algebras is said to be C^∞-morphism if
$$f(\omega(a_1, \ldots, a_n)) = \omega(f(a_1), \ldots, f(a_n)).$$
C^∞- algebras as objects and C^∞-morphisms as morphisms form a category, denotedby C^∞.

Let us notice that $C^\infty(M)$ is an C^∞-algebra.

3. Ringed Space $(\bar{M}_L(A), C^\infty(\bar{M}_L(A))$

In this section, we make a key move for our subsequent considerations. If $(M, C^\infty(M))$ is a differential manifold (or a differential space), and A some algebra, we define a space whose points are maps from M to A, and a differential structure on this space (which turns out to be a C^∞-algebra). It turns out that this structure is isomorphic to the initial differential structure $C^\infty(M)$. Thanks to this, we can use it not only to build the standard geometry of M, but also by substituting various other algebras for A (treating them as stages in the appropriate category), we can construct completely new geometries.

Let $(M, C^\infty(M))$ be an n-dimensional differential manifold (or a differential space), and A an m-dimensional linear space such that $\mathbb{R} \subset A$. By $\bar{M}_L(A) := [C^\infty(M), A]_L$ we will denote the set of linear maps

$$\rho : C^\infty(M) \to A.$$

$\bar{M}_L(A)$ can be interpreted as a functor from the category of C^∞-algebras to the category of linear spaces.

For every smooth function $f \in C^\infty(M)$ we define the function $\bar{f} : \bar{M}_L(A) \to A$ by

$$\bar{f}(\rho) = \rho(f)$$

for any $\rho \in \bar{M}_L(A)$.

It is easy to see that the evaluation $ev_p : C^\infty(M) \to \mathbb{R}$, $ev_p(f) = f(p)$, for any $p \in M$, belongs to $\bar{M}_L(A)$. The set of all functions $\bar{f}$, for $f \in C^\infty(M)$, will be denoted by $C^\infty(\bar{M}_L(A)) = \{\bar{f} : f \in C^\infty(M)\}$.

Lemma 1 *The mapping $J : C^\infty(M) \to C^\infty(\bar{M}_L(A))$ given by*

$$J(f) = \bar{f}$$

is a bijection.

Proof. Suppose $\bar{f} = \bar{g}$ for some $f, g \in C^\infty(M)$. This implies that $\bar{f}(ev_p) = \bar{g}(ev_p)$ for any $p \in M$. Therefore, $f(p) = g(p)$. And finally, $f = g$. $\square$.

Theorem 1 *$C^\infty(\bar{M}_L(A))$ is a C^∞-algebra with the operation of compositions with C^∞-functions*

$$\omega(\bar{f}_1, \ldots, \bar{f}_n) = \overline{\omega(f_1, \ldots, f_n)}$$

for $\omega \in C^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}, f_1, \ldots, f_n \in C^\infty(M)$.

Proof. We check conditions of the definition.

1. It is easy to see that the addition and multiplicaion are of the form

   $$\bar{f}_1 + \bar{f}_2 = \varphi(f_1, f_2)$$

   for $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x_1, x_2) = x_1 + x_2$, and

   $$\bar{f}_1 \cdot \bar{f}_2 = \overline{\psi(f_1, f_2)}$$

   for $\psi : \mathbb{R}^2 \to \mathbb{R}$, $\psi(x_1, x_2) = x_1 \cdot x_2$.
2. There is compatibility of C^∞-algebraic operations with the projections: for $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $\pi_i(x_1, \ldots, x_n) = x_i$, $i = 1, \ldots, n$, $x_1, \ldots, x_n \in \mathbb{R}$, we have

   $$\pi_i(\bar{f}_1, \ldots, \bar{f}_n) = \overline{\pi_i(f_1, \ldots, f_n)} = \bar{f}_i,$$

   $\bar{f}_1, \ldots, \bar{f}_n \in C^\infty(\bar{M}_L(A))$.
3. For $1 \in C^\infty(\mathbb{R}^n)$, $\bar{f}_1, \ldots, \bar{f}_n \in C^\infty(\bar{M}_L(A))$ we have

   $$1(\bar{f}_1, \ldots, \bar{f}_n) = \overline{1(f_1, \ldots, f_n)} = \bar{1}_M \cdot \bar{f} = \overline{1_M \cdot f} = \bar{f}.$$

4. For $\theta \in C^\infty(\mathbb{R}^n)$, $\omega_1, \ldots, \omega_m \in C^\infty(\mathbb{R}^m)$ we have

$$\Theta \circ (\omega_1, \ldots, \omega_m)(\bar{f}_1, \ldots, \bar{f}_n) = \theta(\omega_1(\bar{f}_1, \ldots, \bar{f}_n), \ldots, \omega_m(\bar{f}_1, \ldots, \bar{f}_n)). \ \square$$

Let us notice that addition as define above is a pointwise operation. Indeed,

$$(\bar{f}_1 + \bar{f}_2)(\rho) = (\overline{f_1 + f_2})(\rho) = \rho(f_1 + f_2) = \rho(f_1) + \rho(f_2) = \bar{f}_1(\rho) + \bar{f}_2(\rho)$$

for every $\rho \in \bar{M}_L(A)$. However, multiplication as defined above is not a pointwise operation. Indeed,

$$(\bar{f}_1 \cdot \bar{f}_2)(\rho) = (\overline{f_1 f_2})(\rho) = \rho(f_1 f_2).$$

If ρ is linear but not multiplicative, i.e. if $(f_1 f_2)(\rho) = \rho(f_1)\rho(f_2)$ does not hold, we cannot continue transforming the right-hand side of the above equality. The multiplication operation is pointwise only if ρ is multiplicative, i.e. when we change from the category of linear spaces to the category of C^∞-algebras. In such a case,

$$(\bar{f}_1 \cdot \bar{f}_2)(\rho) = \rho(f_1 f_2) = \rho(f_1)\rho(f_2) = \bar{f}_1(\rho) \cdot \bar{f}_2(\rho).$$

In this way, we obtain the ringed space $(\bar{M}_L(A), C^\infty(\bar{M}_L(A))$ which can be considered as a generalized differential space (in which case we will decorate M with the subscript L). If A is a C^∞-algebra, the linear mappings $\rho : C^\infty(M) \to A$ have the multiplicativity property (in which case we will drop the subscript L). In this case, the corresponding differential space $(\bar{M}(A), C^\infty(\bar{M}(A))$ is a differential subspace of $(\bar{M}_L(A), C^\infty(\bar{M}_L(A))$.

4. Generating Differential Structure of a Functorial Space

In this section, we deal with the issue of generating differential structures for functorial differential spaces. The issue is not obvious because in the case of a functorial space its structure depends on an algebra A. However, we will show that, despite this, the usual method of generating differential structure is nicely transferred to functorial spaces.

Let us recall that the structure of a differential space (M, C) is generated by a subset $C_0 \subset \mathbb{R}^M$ if $C = (scC_0)_M$, and on M we have the topology τ_{C_0}, the weakest topology on M in which the functions from C_0 are continuous. The following Lemma says how to transfer this definition to functorial spaces.

Lemma 2 *If C is generated by C_0 then $\bar{C}(A)$ is generated by $\bar{C}_0(A) = \{\bar{f}^A : f \in C_0\}$.*

Proof. We will show this Lemma locally, for any $U \in \tau_{C_0}$, and for any A (τ_{C_0} is here the weakest topology in which functions of C are continuous), We have

$$\forall_{\rho \in \bar{U}(A)}\, \rho(f|U) = \rho(\omega(g_1, \ldots, g_n)|U \Rightarrow \forall_{\rho \in \bar{U}(A)} \bar{f}(\rho) = \omega(\bar{g}_1, \ldots, \bar{g}_n)(\rho).$$

Therefore, $\bar{f}|\bar{U}(A) = \omega(\bar{g}_1, \ldots, \bar{g}_n)|\bar{U}(A)$. $\square$

We can see that if we have a ringed space $(\bar{M}, \bar{C})$ then $\bar{C}$ is generated by $\bar{C}_0$ if

- $sc\bar{C}_0 = \bar{C}$,
- $(\bar{C}_0)_M = \bar{C}$ (in the sense of sheaf).

Analogous results can be obtained when we allow for linear transformations, i.e. when we consider a ringed space $(\bar{M}_L(A), \bar{C}_L(A))$ with $\bar{M}_L(A) = [C, A]_L$ and $\bar{C}_L(A) = \{\bar{f}_L : f \in C,\ \bar{f}_L(\rho) = \rho(f)$ for $\rho \in \bar{M}_L(A)\}$.

5. Functorial Geometry

In this section, we construct differential geometry for the space $(\bar{M}_L(A), C^\infty(\bar{M}_L(A)))$. From Lemma 1 we know that the C^∞-algebras $C^\infty(M)$ and $C^\infty(\bar{M}_L(A))$ are isomorphic. Therefore, to every derivation $X \in \mathrm{Der}(C^\infty(M))$, there uniquely corresponds the derivation $\bar{X} \in \mathrm{Der}(C^\infty(\bar{M}_L(A)))$ given by $\bar{X}\bar{f} = \overline{Xf}$ for $f \in C^\infty(M)$.

To every metric tensor g on M there uniquely corresponds the metric tensor $\bar{g} : \mathrm{Der}(C^\infty(\bar{M}_L(A))) \times \mathrm{Der}(C^\infty(\bar{M}_L(A))) \to C^\infty(\bar{M}_L(A))$ given by $\bar{g}(\bar{X}, \bar{Y}) = \overline{g(X, Y)}$ for $X, Y \in \mathrm{Der}(C^\infty(M))$.

To every Levi-Civita connection ∇ of a metric tensor g on M there uniquely corresponds the connection $\bar{\nabla} : \mathrm{Der}(C^\infty(\bar{M}_L(A))) \times \mathrm{Der}(C^\infty(\bar{M}_L(A))) \to \mathrm{Der}(C^\infty(\bar{M}_L(A)))$ given by $\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y}$ for $\bar{X}, \bar{Y} \in \mathrm{Der}(C^\infty(\bar{M}_L(A)))$. It satisfies the following conditions

1. $\bar{Z}(\bar{g}(\bar{X}, \bar{Y})) = \bar{g}(\bar{\nabla}_{\bar{Z}}\bar{X}, \bar{Y}) + \bar{g}(\bar{X}, \bar{\nabla}_{\bar{Z}}\bar{Y})$,
2. $[\bar{X}, \bar{Y}] = \bar{\nabla}_{\bar{X}}, \bar{Y} - \bar{\nabla}_{\bar{Y}}, \bar{X}$

for all $\bar{X}, \bar{Y}, \bar{Z} \in \mathrm{Der}(C^\infty(\bar{M}_L(A)))$.

To every curvature tensor $R(X, Y)Z$ of connection ∇ there uniquely corresponds curvature tensor $\bar{R}(\bar{X}, \bar{Y})\bar{Z}$ for connection $\bar{\nabla}$ such that $\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \overline{R(X, Y)Z}$ for any $\bar{X}, \bar{Y}, \bar{Z} \in \mathrm{Der}(C^\infty(\bar{M}_L(A)))$. The following relationships are satisfied

1. $\bar{R}(\bar{X}, \bar{Y})\bar{Z} = -\bar{R}(\bar{Y}, \bar{X})\bar{Z}$,
2. $\bar{g}(\bar{R}(\bar{X}, \bar{Y})\bar{Z}, \bar{U}) = -\bar{g}(\bar{R}(\bar{X}, \bar{Y})\bar{U}, \bar{Z})$,
3. $\bar{R}(\bar{X}, \bar{Y})\bar{Z} + \bar{R}(\bar{Y}, \bar{Z})\bar{X} + \bar{R}(\bar{Z}, \bar{X})\bar{Y} = 0$,
4. $\bar{g}(\bar{R}(\bar{X}, \bar{Y})\bar{Z}, \bar{W}) = \bar{g}(\bar{R}(\bar{Z}, \bar{W})\bar{X}, \bar{Y})$.

In the local basis $\frac{\bar{\partial}}{\partial x^i}$, $i = 1, \ldots, n$, the curvature tensor $\bar{R}$ has the form

$$\bar{R}\Big(\frac{\bar{\partial}}{\partial x^i}, \frac{\bar{\partial}}{\partial x^j}\Big)\frac{\bar{\partial}}{\partial x^k} = \sum_{l-1}^{n} \bar{R}^l_{ijk}\frac{\bar{\partial}}{\partial x^l}.$$

The trace of a tensor $\bar{B}$ of type $(1, 1)$

$$\bar{B} : \mathrm{Der}(C^\infty(\bar{M}_L(A))) \to \mathrm{Der}(C^\infty(\bar{M}_L(A)))$$

has locally the form

$$\mathrm{tr}\bar{B} = \mathrm{tr}(\bar{B}^j_i)$$

where $\bar{B}(\frac{\bar{\partial}}{\partial x^i}) = \sum_{j=1}^n \bar{B}^j_i \frac{\bar{\partial}}{\partial x^j}$. Obviously, one has $\mathrm{tr}\bar{B} = \overline{\mathrm{tr}B}$.

Let us consider two tensors $R(X, Y)$, $X, Y \in \mathrm{Der}(C^\infty(M)$ and a tensor $\bar{R}(\bar{X}, \bar{Y})$, $\bar{X}, \bar{Y} \in \mathrm{Der}(C^\infty(\bar{M}_L(A))$ remaining in one-to-one correspondence. One has $\bar{R}_{\bar{X},\bar{Y}} = \overline{R_{X,Y}}$.

As well known, Ricci tensor satisfies the following relation $\mathrm{Ric}(X, Y) = \mathrm{tr}R_{X,Y}$ and correspondingly $\overline{\mathrm{Ric}}(\bar{X}, \bar{Y}) = \mathrm{tr}\bar{R}(\bar{X}, \bar{Y}) = \overline{\mathrm{tr}R_{X,Y}}$.

We know that there exists an operator $\mathcal{R} : \mathrm{Der}(C^\infty(M)) \to \mathrm{Der}(C^\infty(M))$ such that $\mathbf{R}ic(X, Y) = g(\mathcal{R}X, Y)$ where g is a Lorentz metric on $\mathrm{Der}(C^\infty(M))$. Correspondingly, we have an operator $\bar{\mathcal{R}} : \mathrm{Der}(C^\infty(\bar{M}_L(A)) \to \mathrm{Der}(C^\infty(\bar{M}_L(A))$ such that $\mathbf{R}ic(\bar{X}, \bar{Y}) = \bar{g}(\bar{\mathcal{R}}\bar{X}, \bar{Y}) = \overline{g(\mathcal{R}X, Y)}$ where $\bar{g}$ is a Lorentz metric on $\mathrm{Der}(C^\infty(\bar{M}_L(A))$. Then the scalar (or Ricci) curvature is defined as $\bar{r} = \mathrm{tr}\bar{\mathcal{R}}$.

To have a clear picture of the above, let us notice that our starting manifold (or a differential space) $(M, C^\infty(M))$ is embedded into $(\bar{M}_L(A), C^\infty(\bar{M}_L(A))$. Indeed,

$$(M, C^\infty(M)) \cong ((\bar{M}_{ev}(A), C^\infty(\bar{M}_{ev}(A))) \subset (\bar{M}_L(A), C^\infty(\bar{M}_L(A))).$$

The space $\bar{M}_L(A)$ contains $\bar{M}_{ev}(A)$ as a space of linear functionals $\rho : C^\infty(M) \to A$. In particular, evaluations $ev_p : C^\infty(M) \to \mathbb{R} \subset A$ can be interpreted as original points of the functional space $\bar{M}_L(A)$.

As we can see, we already have all the quantities necessary to formulate the functorial Einstein algebra.

6. Functorial Einstein Algebra

The idea of Einstein algebra comes from Geroch (Geroch 1972); it was later developed in (Heller 1992) and (Heller and Sasin 1995). In (Heller et al. 2024), we defined the Lorentz module on a differential manifold (or a differential space) (M, g) as a triple $(\mathrm{Der}(C^\infty(M)), C^\infty(M), g)$, where $\mathrm{Der}C^\infty(M)$ is a $C^\infty(M)$-module of derivations of the algebra $C^\infty(M)$, and the Einstein algebra on (M, g) as a Lorentz module on (M, g) on which the Einstein equations are defined (although there the definition was given a slightly different wording). The functorial version of this definition has the following form

Definition 2 • *Functorial Lorentz module on a functorial Lorentz manifold* $(\bar{M}_L(A), \bar{g})$ *is a triple* $(\mathrm{Der}(C^\infty(\bar{M}_L(A))), C^\infty(\bar{M}_L(A)), \bar{g})$*, where* $\mathrm{Der}(C^\infty(M))$ *is a* $C^\infty(M)$*-module of derivations of the algebra* $C^\infty(\bar{M}_L(A))$*.*

- *Einstein algebra on a functorial Lorentz manifold* $(\bar{M}_L(A), \bar{g})$ *is a functorial Lorentz module* $(\mathrm{Der}(C^\infty(\bar{M}_L(A))), C^\infty(\bar{M}_L(A)), \bar{g})$ *on which functorial Einstein equations are defined. They are of the form*

$$(i)\ \ \overline{\mathbf{Ric}} = \Lambda\bar{g},$$

$$(ii)\ \ \overline{\mathbf{Ein}} + \Lambda\bar{g} = 8\pi\bar{T}$$

where $\overline{\mathbf{Ein}} = \overline{\mathbf{Ric}} - \frac{1}{2}\bar{r}\bar{g}$ *is called the Enstein tensor,* Λ *is the cosmological constant, and* $\bar{T}$ *is a suitable energy-momentum tensor.*

Sometimes, unless there is danger of confusion, only the expression $C^\infty(\bar{M}(A))$ will be called Einstein algebra. The above functorial (generalized) Einstein equations select from the $C^\infty(\bar{M}_L(A))$-module of derivations $\mathrm{Der}(C^\infty((\bar{M}_L(A))))$ a submodule of derivations satisfying these equations. The Lorentz metric $\bar{g}$ is defined on this submodule. In ordinary general relativity, the dependence on the derivation submodule remains invisible because the full derivation module is assumed to be considered from the very beginning. In this case, Einstein's equations determine only the Lorentz metric.

Let us notice that if $A = \mathbb{R}$, we have $\rho = ev_p : C^\infty(M) \to \mathbb{R}$ given by $ev_p(f) = f(p)$ and the functorial differential space becomes the usual differential space $(M, C^\infty(M))$. The fact that assuming A different from $\mathbb{R}$ may be interesting is demonstrated by an example: The assumption that A is a Grassmann algebra leads to the theory of supermanifolds (see Section 8).

7. Functorial Structured Spaces

According to the method adopted in the previous chapters of these notes, after developing the differential version of the theory of given spaces, one should move to its structured version. In the case of functorial spaces, this transition is natural. The structured counterpart of functorial spaces is the sheaf of differential structures of functorial spaces over the topological space $(M, \mathrm{top}M)$.

We construct such a sheaf in the following way.

1. With every open subset $U \in \mathrm{top}M$ we associate a functorial space $(\bar{U}_L, C^\infty(\bar{U}_L))$.
2. For any two open subsets $U, V \in \mathrm{top}M$ such that $U \subset V$, the restriction operator is defined $\rho^V_U : C^\infty(\bar{V}_L) \to C^\infty(\bar{U}_L)$ by $\rho^V_U(\bar{f}_L) = \bar{f}|U, \bar{f}_L \in C^\infty(\bar{V})$ for $f \in C^\infty(M)$.

Definition 3 *The functorial structured space over a topological space* $(M, \mathrm{top}M)$ *is a pair* $(M, \mathcal{O}^L_M(A))$ *where* $\mathcal{O}^L_M(A)$ *is a sheaf over* $(M, \mathrm{top}M)$ *such that for every* $U \in \mathrm{top}M$ *one has* $\mathcal{O}^L_M(A)(U) = C^\infty(\bar{U}_L(A))$.

Obviously, if A is a C^∞-algebra, the points ρ enjoy the multiplicativity property and the subscript L can be omitted everywhere.

8. Example: Functorial Approach to Supermanifolds

In this section, we show that if the differential structure of a functorial manifold (differential space) is tensor multiplied by a Grassmann algebra, one obtains a functorial supermanifold (functorial super differential space). A Grassmann algebra is an associative $\mathbb{R}$-algebra Λ_n with unity, such that there exists a set of generators in it $1, \beta_1, \ldots, \beta_n$, $n \in \mathbb{N}$, such that

$$1_{\Lambda_n}\beta_i = \beta_i = \beta_i 1_{\Lambda_n}$$

and

$$\beta_i\beta_k + \beta_k\beta_i = 0.$$

Every element $\alpha \in \Lambda_n$ can be written as

$$\alpha = \sum_{k\geq 0}\sum_{i_1,\ldots,i_k} \alpha_{i_1,\ldots,i_k}\beta_{i_1}\ldots\beta_{i_k},$$

Indices $i_1, \ldots, i_k$ are supposed to satisfy one of the following conditions: either $\alpha_{i_1,\ldots,i_k}$ should be antisymmetric with respect to their indices, or the indices should form the increasing sequence $i_1 < i_2 < \ldots < i_k$. Obviously, β_i are nilpotents ($\beta_i^2 = 0$). The above definition of Grassmann algebra can be rephrased in such a way as to allow an infinite number of generators, but we will not deal here with this case.

A unique algebra homomorphism $b : \Lambda_n \to \mathbb{R}$ which maps 1 onto 1 and all the generators to zero is called the body map and its image the body of Λ_n. The linear map s which maps Λ_n onto its nilpotent elements is called the soul map and its image the soul of Λ_n.

Any Grassmann algebra admits a gradation $\Lambda_n = {}^0\Lambda_n \oplus {}^1\Lambda_n$ where ${}^0\Lambda_n$ and ${}^1\Lambda_n$ are spanned by the products of even and odd numbers of generators, respectively. The subspace ${}^0\Lambda_n$ is also a subalgebra of Λ_n. (For more on Grassmann algebras see (Berezin 1983; Berezin and Leites 1975; DeWitt 1984; Rogers 2007).)

There are at least several (not all equivalent) definitions of a supermanifold: for instance, Berezin-Leites original definition (Berezin 1983; Berezin and Leites 1975), Kostant's definition (Kostant 1977), DeWitt's definition in terms of maps and atlases (DeWitt 1984). In what follows, we will refer to Berezin's definition (as it was formulated by Rogers (Rogers 2007, p. 86) for its simplicity. The definition runs as follows.

Definition 4 *A smooth real (algebro-geometric) supermanifold of dimension* (m, n) *is a pair* $(M, \mathcal{A})$ *where* M *is a real* m*-dimensional manifold and* $\mathcal{A}$ *a sheaf of supercommutative algebras over* M *satisfying the following conditions*

1. *there exists an open cover* $\{U_i | U_i \in \mathrm{top}M, i \in I\}$ *of* M *such that for each* $i \in I$

$$\mathcal{A}(U_i) \cong C^\infty(U_i) \otimes \Lambda_n,$$

   where Λ_n *is a Grassmann algebra with* n *generators,*

2. *if* $\mathcal{N}$ *is the sheaf of nilpotents in* $\mathcal{A}$*, then*

$$(M, \mathcal{A}/\mathcal{N}) \cong (M, C^\infty_M)$$

   where C^∞_M *is a sheaf of smooth functions on* M.

We will call a supermanifold defined in this way the Berezin supermanifold. Let us notice that an $(m, 0)$-dimensional supermanifold is an m-dimensional Berezin manifold.

Let us now consider a functorial differential space with the differential structure tensorially multiplied by a Grassmann algebra, i.e. $(\bar{M}(A), C^\infty(\bar{M}(A) \otimes \Lambda_n))$, which we change, in the standard way, into the sheaf space $(\bar{M}(A), \mathcal{A}_{\bar{M}})$ with $\mathcal{A}_{\bar{M}}(\bar{U}_i) \cong C^\infty(\bar{U}_i) \otimes \Lambda_n$. We will consider this construction in the *Lin* or C^∞ categories, as appropriate.

Let us first notice that multiplication by a tensor does not spoil the smoothness. The mapping

$$\bar{F} : (\bar{M}(A), C^\infty(\bar{M}(A))) \to (\bar{N}(A), C^\infty(\bar{N}(A))),$$

given by $\bar{F}(\rho) = \rho \circ F^*$, $\rho \in \bar{M}(A)$, is smooth, if for every $\beta \in C^\infty(N)$, one has $\bar{\beta} \circ \bar{F} \in C^\infty(\bar{M}(A))$. It can be easily checked that in our case

$$\bar{F} : (\bar{M}(A), C^\infty(\bar{M}(A)) \otimes \Lambda_n) \to (\bar{N}(A), C^\infty(\bar{N}(A)) \otimes \Lambda_n),$$

this is satisfied. Indeed,

$$(\bar{\beta} \otimes \lambda) \circ \bar{F} = \overline{\beta \circ F} \otimes \Lambda_n,$$

$\lambda \in A$.

The tensor multiplication of the differential structure $C^\infty(\bar{M}_L(A))$ by A has one more consequence: it breaks the isomorphism between the $C^\infty(M)$ and $C^\infty(\bar{M}_L(A))$ structures, thus enriching the geometry of the latter.

Now, let us consider a smooth supermanifold of dimension (m, n) as a pair $(M, \mathcal{A})$ where M is a real m-dimensional manifold and $\mathcal{A}$ a sheaf of supercommutative algebras as in Definition 4. On the strength of the isomorphism

$$J : C^\infty(\bar{U}(A)) \otimes \Lambda_n \to C^\infty(U) \otimes \Lambda_n$$

given by

$$J(\bar{f} \otimes \lambda) = f \otimes \lambda,$$

$\lambda \in A$, we can associate (uniquely) with each Berezin supermanifold $(M, \mathcal{A})$, where, for each $i \in I$, $\mathcal{A}(U_i) \cong C^\infty(U_i) \otimes \Lambda_n$, the functorial space $(\bar{M}(A), \bar{\mathcal{A}})$, where, for each $i \in I$, $\bar{\mathcal{A}}(\bar{U}_i) \cong C^\infty(\bar{U}_i) \otimes \Lambda_n$ (the second condition of Definition 4 is met automatically). The latter space can also be called (functorial) Berezin supermanifold or extended Berezin supermanifold. (Of course, for A we can also substitute Λ_n.)

The concept of Einstein-Grassmann algebra from the end of Section 7 is automatically transferred to the current situation. We will call the sheaf $\bar{\mathcal{A}}$ the Einstein-Grassmann sheaf if, for every $\bar{U}_i$, $\bar{\mathcal{A}}_{\bar{U}_i} \cong C^\infty(\bar{U}_i) \otimes \Lambda_n$, the algebra $C^\infty(\bar{U}_i)$ is an Einstein algebra. A supermanifold with an Einstein-Grassmann algebra as its differential structure will be called Einstein-Grassmann supermanifold.

In (Heller et al. 2024), we investigated a number of properties of functorial supermanifolds. In particular, curves in supermanifolds (super curves) exhibit rich properties which curves in an ordinary manifold are lacking. This may have important implications for the theory of singularities in superspace.

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3. Functorial Lie Groups

Ewa Falkiewicz and Wiesław Sasin

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Functorial Lie Groups

E. Falkiewicz*†‡ and W. Sasin†

†Institute of Mathematics and Cryptology, Faculty of Cybernetics, Military University of Technology, Warsaw, 00-908, Gen. Sylwestra Kaliskiego 2, Poland
‡Chair of International Business and Finance, Faculty of Economics and Finance, Kazimir Pulaski Radom University, 26-600 Radom, Chrobrego 31, Poland
*Corresponding author. Email: ewa.falkiewicz@wat.edu.pl

Abstract
In this paper we present the concept of a functorial Lie group. We enrich the space with parametric points and consider the structure as C^∞ – algebra [Pysiak et al. 2020]. We get varied spaces. Taking the Lie group as a space, we construct the functorial Lie group with parametric points. We also consider the Lie algebra of the functorial Lie group and construct the functorial exponential map.

Keywords: Lie group, Lie algebra, left – invariant vector field, parametric points, functorial Lie group, exponential map

1. Preliminaries

Let $(G, C^\infty(G))$ be any Lie group, i.e. a group which also has the structure of a manifold, with the group operations:

$$\begin{aligned} \mu : G \times G \to G, \quad \mu(g_1, g_2) = g_1 g_2, \\ \nu : G \to G, \quad \nu(g) = g^{-1}, \end{aligned} \tag{1}$$

for any $g_1, g_2, g \in G$ and the neutral element $e \in G$.

For $a \in G$ we define mappings: the left translation $L_a : G \to G$ in the group G by the element a, $L_a(g) = ag$, the right translation $R_a : G \to G$ in the group G by the element a, $R_a(g) = ga$, and the inner automorphism $\mathrm{ad}_a : G \to G$, $\mathrm{ad}_a(g) = aga^{-1}$.

The mappings R_a and L_a are diffeomorphisms. They induce differential maps (the push-forwards) $(R_a)_{*g} : T_gG \to T_{ga}G$ and $(L_a)_{*g} : T_gG \to T_{ag}G$ respectively, which are given by the formulas:

$$((R_a)_* v)(f) = v(f \circ R_a),$$

$$((L_a)_* v)(f) = v(f \circ L_a)$$

for any $v \in T_gG$ and $f \in C^\infty(G)$.

Let $X : G \to TG$ be a left – invariant vector field on G, i.e. the vector field which is invariant with respect to any left translation on G:

$$\forall a \in G \quad (L_a)_*(X) = X,$$

or equivalently

$$\forall a \in G \quad (L_a)_* X(e) = X(a).$$

Let us denote by $\mathcal{L}(G)$ the set of all left - invariant vector fields on G,

$$\mathcal{L}(G) = \{X \in \mathcal{X}(G) : \forall a \in G\ (L_a)_*(X) = X\}.$$

$\mathcal{L}(G)$ is a Lie algebra, which is called the *Lie algebra of the Lie group* G. It is the set of all left - invariant vector fields on G with the usual addition, scalar multiplication and bracket operation [Kobayashi and Nomizu 1963].

The mapping $i_G : \mathcal{L}(G) \to T_eG$, $i_G(X) = X(e)$, is an isomorphism of linear spaces [Gancarzewicz 2010]. So it allows us to transfer the Lie bracket $[\ ,]$ from $\mathcal{L}(G)$ to T_eG and

$$\dim \mathcal{L}(G) = \dim T_eG.$$

$\mathcal{L}(G)$ is the Lie subalgebra of the dimension n ($n = \dim G$) of the Lie algebra of all vector fields $\mathcal{X}(G)$. It is known [Kobayashi and Nomizu 1963], that every $X \in \mathcal{L}(G)$ generates a global flow as 1-parameter group of transformations of G, i.e. C^∞ mapping $\varphi : \mathbf{R} \times G \to G$, $\varphi_t : G \to G$, given by $\varphi_t(g) = \varphi(t, g)$, satisfying the conditions:

1. φ_t is diffeomorphism for all $t \in \mathbf{R}$,
2. $\forall_{t,s\in\mathbf{R}}\ \forall_{g\in G}\quad \varphi_t(\varphi_s(g)) = \varphi_{t+s}(g)$,
3. $\forall_{g\in G}\quad \varphi_0(g) = g$.

For fixed point $g \in G$, $\varphi_g(t)$ is the integral curve of the field X starting at $g \in G$. We have

$$X(\varphi_g(t)) = \frac{d\varphi_g(s)}{ds}\Big|_{s=t} = \varphi_g'(t).$$

Theorem 1.1 *If $X \in \mathcal{L}(G)$ and φ_t is the global flow of X, then for $a_t := \varphi_t(e)$, $t \in \mathbf{R}$, we have the following formulas [Gancarzewicz 2010]:*

1. $a_t \cdot a_s = a_{t+s}$,
2. $a_0 = \varphi_0(e) = e$,
3. $\varphi_t = R_{a_t}$.

The family $\{a_t\}_{t\in\mathbf{R}}$ is called *the 1-parameter subgroup of G generated by an element $X \in \mathcal{L}(G)$*. In another characterisation a_t is a unique curve in G such that its tangent vector $\dot{a}_t$ at a_t is equal to $L_{a_t}(X(e))$. In other words, a_t is a unique solution of the differential equation

$$a_t^{-1}\dot{a}_t = X(e)$$

with initial condition $a_0 = e$.
Theorem 1.1 allows us to define a mapping

$$\exp_G : \mathcal{L}(G) \to G,$$

$$\exp_G(A) = a_1 = \varphi_1(e),$$

which is called the *exponential mapping*.
The exponential mapping $\exp_G : \mathcal{L}(G) \to G$ is smooth and has the following properties:

Theorem 1.2 *If $\{a_t\}_{t\in\mathbf{R}}$ is the 1-parameter subgroup of G generated by an element $A \in \mathcal{L}(G)$, then:*

1. *for any $s \in \mathbf{R}$, $\exp_G(sA) = a_s$,*
2. *for $k \in \mathbf{Z}$, $\exp_G(kA) = (\exp_G(A))^k$,*
3. $\exp_G(0) = a_0 = e$.

2. Space with parametric points

Let $(M, C^\infty(M))$ be a differential manifold, $(P, C^\infty(P))$ be a manifold of parameters. Every mapping $\varphi : P \to M$ is called *the parametric point of the manifold M.* Spaces with parametric points were considered in [Navarro Gonzalez and Salas J. B. 2003], [Pysiak et al. 2020], [Falkiewicz, E. and Sasin, W. 2021].

By $\bar{M}(P)$ we denote the set of all parametric points $\varphi : P \to M$ of the manifold M. Let us notice that

$$\bar{M}(P) = M^P.$$

For any smooth function $f \in C^\infty(M)$ we define $\bar{f} : \bar{M}(P) \to C^\infty(P)$ by the formula

$$\bar{f}(\varphi) = f \circ \varphi \quad \text{for} \quad \varphi \in M^P. \tag{2}$$

By the symbol $C^\infty(\bar{M}(P))$ we denote the set of functions $\bar{f}$, where $f \in C^\infty(M)$,

$$C^\infty(\bar{M}(P)) = \{\bar{f} : f \in C^\infty(M)\}.$$

It is easy to see that $C^\infty(\bar{M}(P))$ is a C^∞ - algebra with operations

$$\omega(\bar{f}_1, \ldots, \bar{f}_n) = \overline{\omega(f_1, \ldots, f_n)}$$

for any $\omega \in C^\infty(\mathbf{R}^n)$, $n \in \mathbf{N}, f_1, \ldots, f_n \in C^\infty(M)$. We obtain the ringed space $(\bar{M}(P), C^\infty(\bar{M}(P)))$ of parametric points with parameters from P, associated with the manifold $(M, C^\infty(M))$.

For any smooth mapping $F : M \to N$ of manifolds M and N we define the mapping

$$\bar{F} : \bar{M}(P) \to \bar{N}(P)$$

by the formula

$$\bar{F}(\varphi) = F \circ \varphi.$$

The mapping $\bar{F}$ is smooth, $\bar{F} : (\bar{M}(P), C^\infty(\bar{M}(P))) \to (\bar{N}(P), C^\infty(\bar{N}(P)))$. Indeed,

$$\bar{F}^*(\bar{\beta}) = \overline{F^*(\beta)} = \overline{\beta \circ F} \quad \text{for} \quad \beta \in C^\infty(N).$$

Lemma 2.1 *The mapping $J : C^\infty(M) \to C^\infty(\bar{M}(P))$ given by*

$$J(f) = \bar{f}$$

is an isomorphism of the C^∞ - rings.

Proof: If $\bar{f} = \bar{g}$ then $\forall \varphi \in M^P$ we have $\bar{f}(\varphi) = \bar{g}(\varphi)$, i.e. $f \circ \varphi = g \circ \varphi$. Then $\forall p \in P$, $f \circ \varphi(p) = g \circ \varphi(p)$ or otherwise $f(\varphi(p)) = g(\varphi(p))$. From this we obtain $f = g$ $\forall x = \varphi(p) \in M$. QED

More generally, we construct a functor

$$\bar{M} : \mathbf{Diff} \to \mathbf{Sets},$$

where **Diff** is a category of differential manifolds and **Sets** is a category of sets. At a stage $P \in \mathbf{Diff}$ we have

$$\bar{M}(P) = M^P.$$

For any function $f \in C^\infty(M)$ we define 1-parameter family of natural transformations as follows:

$$\bar{f} := \{\bar{f}^P\}_{P \in \mathbf{Diff}},$$

At a stage P the mapping $\bar{f}^P : \bar{M}(P) \to C^\infty(P)$ is defined by (2), $\bar{f}^P(\varphi) = f \circ \varphi$.

We obtain the ringed space $(\bar{M}, C^\infty(\bar{M})$ with the C^∞ - algebra $C^\infty(\bar{M})$ isomorphic to the C^∞ - algebra $C^\infty(M)$ (see [Palais 1981], [Sikorski 1972], [Nestruev 2003]).

3. Lie groups with parametric points

Let $(G, C^\infty(G))$ be the Lie group with the actions (1). Let us consider space $(\bar{G}(P), C^\infty(\bar{G}(P)))$ of parametric points $\varphi : P \to G$, associated with the manifold $(G, C^\infty(G))$.
The smooth mappings $\bar{\mu} : \bar{G}(P) \times \bar{G}(P) \to \bar{G}(P)$ and $\bar{\nu} : \bar{G}(P) \to \bar{G}(P)$ are group actions in $\bar{G}(P)$. We obtain *differential Lie group* $(\bar{G}(P), C^\infty(\bar{G}(P)))$ with group actions

$$\varphi_1 \cdot \varphi_2 = \bar{\mu}(\varphi_1, \varphi_2) \text{ and } \varphi^{-1} = \bar{\nu}(\varphi)$$

for any parametric points $\varphi, \varphi_2, \varphi \in \bar{G}(P) = G^P$. The neutral element of $\bar{G}(P)$ is constant mapping

$$\varphi_e : P \to G, \quad \varphi_e(p) = e \text{ for any } p \in P.$$

Obviously

$$\varphi_1 \cdot \varphi_2 = \bar{\mu}(\varphi_1, \varphi_2) = \mu \circ (\varphi_1, \varphi_2) = \mu(\varphi_1, \varphi_2),$$
$$\varphi^{-1} = \bar{\nu}(\varphi) = \nu \circ \varphi.$$

From the above

$$(\varphi_1, \varphi_2)(p) = \varphi_1(p) \cdot \varphi_2(p), \quad \varphi^{-1}(p) = (\varphi(p))^{-1} \text{ for all } p \in P.$$

Proposition 3.1 $(\bar{G}(P), C^\infty(\bar{G}(P)))$ *is the Lie group with parametric points, associated with the Lie group* $(G, C^\infty(G))$.

By Lemma 2.1 $C^\infty(\bar{G}(P)) = \{\bar{f} : \bar{f}(\varphi) = f \circ \varphi, \varphi \in G^P\}$ is a C^∞ – algebra isomorphic to $C^\infty(G), f \mapsto \bar{f}$ is an isomorphism. From this fact we obtain that any derivation $X \in \mathrm{Der}(C^\infty(G))$, $X : C^\infty(G) \to C^\infty(G)$ induces the derivation $\bar{X} \in \mathrm{Der}(C^\infty(\bar{G}))$:

$$\bar{X}\bar{f} = \overline{Xf}.$$

The extension $\bar{X}$ of a derivation X is unique, i.e. if $\bar{X} = \bar{Y}$ then $X = Y$.
If we take the constant mapping $\epsilon : P \to G$, where $\epsilon(p) = e$ for all $p \in P$, then we obtain

$$\bar{X}(\epsilon) = X \circ \epsilon = X(e).$$

We say that the derivation $\bar{X} \in \mathrm{Der}(C^\infty(\bar{G}))$, $\bar{X} : C^\infty(\bar{G}) \to C^\infty(\bar{G})$, is left-invariant, if for any $\varphi \in G^P$,

$$(L_\varphi)_* \bar{X}(\epsilon) = \bar{X}(\varphi). \tag{3}$$

Let us denote by $\mathcal{L}(\bar{G})$ the set of all left-invariant derivations $\bar{X}$ of the algebra $C^\infty(\bar{G})$. $\mathcal{L}(\bar{G})$ is the Lie algebra with the bracket operation $[\cdot, \cdot] : C^\infty(\bar{G}) \times C^\infty(\bar{G}) \to C^\infty(\bar{G})$,

$$[\bar{X}, \bar{Y}] = \bar{X} \circ \bar{Y} - \bar{Y} \circ \bar{X} = \overline{X \circ Y - Y \circ X} = \overline{[X, Y]}.$$

The algebra $\mathcal{L}(\bar{G})$ will be called *the Lie algebra of the functorial Lie group* $\bar{G}$.
On a stage P we have $\mathcal{L}(G^P)$ – the Lie algebra of the Lie group G^P of parametric points.

Proposition 3.2 *Any left-invariant derivation $X \in \mathcal{L}(G)$ induces the left-invariant derivation $\bar{X} \in \mathcal{L}(\bar{G})$.*

Sketch of the proof. Analogously like in a classical case, each left-invariant vector field $\bar{X} \in \mathcal{L}(\bar{G})$ induces its global flow $\bar{\varphi} : \bar{\mathbf{R}} \times \bar{G}(P) \to \bar{G}(P)$ (otherwise $\bar{\varphi} : \mathbf{R}^P \times G^P \to G^P$) which we define by the formula

$$\bar{\varphi}(\tau, \gamma) = \bar{\varphi}_\tau(\gamma)$$

for fixed $\tau : P \to \mathbf{R}$ and for any parametric point $\gamma \in G^P, \gamma : P \to G$. $\bar{\varphi}_\tau$ is the 1-parameter group of transformations of $\bar{G}(P)$ generated by the left – invariant vector field $\bar{X} \in \mathcal{L}(G^P)$.
$\bar{\varphi}_\tau$ possesses similar properties to classical case:

1. $\bar{\varphi}_\tau$ is diffeomorphism for all $\tau \in \mathbf{R}^P$,
2. $\forall_{\tau_1,\tau_2\in\mathbf{R}^P}\ \forall_{\gamma\in G^P}\quad \bar{\varphi}_{\tau_1}(\bar{\varphi}_{\tau_2}(\gamma)) = \bar{\varphi}_{\tau_1+\tau_2}(\gamma)$,
3. $\forall_{\gamma\in G^P}\quad \bar{\varphi}_\theta(\gamma) = \gamma$ (or equivalently $\bar{\varphi}_\theta = \mathrm{id}_{G^P}$), where $\theta : P \to \mathbf{R}$, $\forall_{p\in P}\ \theta(p) = 0$.

For fixed $\gamma \in \bar{G}(P)$, $\bar{\varphi}_\gamma(\tau)$ is the integral curve of the vector field $\bar{X}$ starting in $\gamma \in \bar{G}(P)$, where $\tau \in \mathbf{R}^P$. We have

$$\bar{X}(\bar{\varphi}_\gamma(\tau)) = \frac{d\bar{\varphi}_\gamma(\sigma)}{d\sigma}\ |_{\sigma=\tau} = \bar{\varphi}'_\gamma(\tau).$$

Lemma 3.1 *If $\bar{X} \in \mathcal{L}(\bar{G})$ is a left-invariant vector field on $\bar{G}(P)$ corresponding to the left-invariant vector field $X \in \mathcal{L}(G)$ on G via isomorphism of C^∞ - algebras $J : C^\infty(G) \to C^\infty(\bar{G}(P))$ (lemma 2.1), then the global flows for both vector fields are bijective, i.e. for any integral curve of the vector field X there is exactly one integral curve of the vector field $\bar{X}$.*

Taking the left-invariant vector field $\bar{X} \in \mathcal{L}(\bar{G})$ and the global flow $\bar{\varphi}_\tau$ of $\bar{X}$, we define

$$\bar{a}_\tau := \bar{\varphi}_\tau(\epsilon) \in G^P,$$

where $\epsilon \in G^P$ is the constant mapping $\epsilon : P \to G$ such that $\epsilon(p) = e$ for all $p \in P$, where e is the neutral element in the Lie group G. $\bar{a}_\tau$ has properties:

1. $\bar{a}_{\tau_1} \cdot \bar{a}_{\tau_2} = \bar{a}_{\tau_1+\tau_2}$ for $\tau_1, \tau_2 \in \mathbf{R}^P$,
2. $\bar{a}_\theta = \bar{\varphi}_\theta(\epsilon) = \epsilon$, for $\theta \in \mathbf{R}^P, \theta(p) = 0$ for all $p \in P$.

The family $\{\bar{a}_\tau\}_{\tau\in\mathbf{R}^P}$ is called *the 1-parameter subgroup of $\bar{G}(P)$ generated by the left - invariant vector field $\bar{X} \in \mathcal{L}(\bar{G}(P))$*.
Further, using $\bar{a}_\tau$, we define the *exponential mapping for the Lie group $\bar{G}(P)$*:

$$\exp_{\bar{G}(P)} : \mathcal{L}(\bar{G}(P)) \to \bar{G}(P),$$

$$\exp_{\bar{G}(P)}(\bar{X}) = \bar{a}_{\mathbf{1}} = \bar{\varphi}_{\mathbf{1}}(\epsilon),$$

where $\mathbf{1} : P \to \mathbf{R}$ is the constant mapping such that $\mathbf{1}(p) = 1$ for each $p \in P$.

Theorem 3.1 *If $\{\bar{a}_\tau\}_{\tau\in\mathbf{R}^P}$ is the 1-parameter subgroup of $\bar{G}(P)$ generated by the element $\bar{X} \in \mathcal{L}(\bar{G}(P))$ for $\tau \in \mathbf{R}^P$, then:*

1. *for any $\sigma \in \mathbf{R}^P$, $\exp_{\bar{G}(P)}(\sigma \cdot \bar{X}) = \bar{a}_\sigma$ (because $\{\bar{a}_{\sigma\cdot\tau}\}$ is the 1-parameter subgroup of $\bar{G}(P)$ generated by the left - invariant vector field $\sigma \cdot \bar{X} \in \mathcal{L}(\bar{G}(P))$),*
2. *for $k \in \mathbf{Z}$ and $\boldsymbol{k} : P \to \mathbf{R}$ such that $\boldsymbol{k}(p) = k$ for any $p \in P$, $\exp_{\bar{G}(P)}(\boldsymbol{k} \cdot \bar{X}) = \bar{a}_{\boldsymbol{k}} = \bar{a}_{\mathbf{1}} \cdot \ldots \cdot \bar{a}_{\mathbf{1}} = (\exp_{\bar{G}(P)}(\bar{X}))^k$,*
3. *$\exp_{\bar{G}(P)}((\tau_1 + \tau_2)\bar{X}) = \exp_{\bar{G}(P)}(\tau_1\bar{X}) \exp_{\bar{G}(P)}(\tau_2\bar{X})$ for any $\tau_1, \tau_2 \in \mathbf{R}^P$,*
4. *$\exp_{\bar{G}(P)}(\theta) = \bar{a}_\theta = \bar{\varphi}_\theta(\epsilon) = \epsilon$.*

4. Classical Lie groups with parametric points

Let us denote by

$$\mathrm{GL}(n, \mathbf{R}) = \{A \in M_n(\mathbf{R}) : \det A \neq 0\}$$

the well known classical linear Lie group of $n \times n$ real, invertible matrices.

Let $(P, C^\infty(P))$ be a differential manifold of parameters. The associated group to $\mathrm{GL}(n, \mathbf{R})$, with parametric points $\varphi : P \to \mathrm{GL}(n, \mathbf{R})$, is of the form:

$$\begin{aligned}\mathrm{GL}(n, \mathbf{R}^P) = \mathrm{GL}(n, C^\infty(P)) \quad &= \{\varphi : P \to \mathrm{GL}(n, \mathbf{R}) : \forall_{p \in P}\ \varphi(p) \in \mathrm{GL}(n, \mathbf{R})\} = \\ &= \{\varphi = (\varphi_{ij}) : \varphi_{ij} \in C^\infty(P), i, j = 1, \ldots, n\}.\end{aligned}$$

The parametric points of the form $\varphi : P \to \mathrm{GL}(n, \mathbf{R})$ of $\mathrm{GL}(n, C^\infty(P))$, are $n \times n$ invertible matrices, such that $\det \varphi \neq \theta$, where $\theta : P \to \mathbf{R}$ is the constant function satisfying the condition $\theta(p) = 0$ for all $p \in P$.
Analogously to each classical Lie group G of $n \times n$ matrices we can assign the group with parametric points:

$$\bar{G}(P) = G^P = \{\varphi : P \to G : \forall_{p \in P}\ \varphi(p) \in G\} = \{\varphi = (\varphi_{ij}) : \varphi_{ij} \in C^\infty(P), i, j = 1, \ldots, n\}.$$

In other words, for each classical Lie group $G(n, \mathbf{R})$ of $n \times n$ real matrices we can assign functorial group $G(n, \overline{\mathbf{R}})$ of $n \times n$ matrices with elements in $\overline{\mathbf{R}}$, where $\overline{\mathbf{R}}$ is real line with parametric points, $\overline{\mathbf{R}}(P) = \mathbf{R}^P = C^\infty(P)$.

For example, taking classical special orthogonal group $G = \mathrm{SO}(n, \mathbf{R})$, we can construct the functorial group

$$\mathrm{SO}(n, \overline{\mathbf{R}}) = \{A \in M_n(\overline{\mathbf{R}}) : AA^T = A^T A = I \wedge \det A = 1\} = \{A = (a_{ij}) : a_{ij} \in \overline{\mathbf{R}}\}.$$

Let us consider a mapping

$$\bar{\psi} : \mathbf{R}^P \to \mathrm{GL}(n, \mathbf{R}^P),$$

given by the following formula

$$\bar{\psi}(\tau) = \sum_{i=0}^{\infty} \frac{1}{i!}(\tau\varphi)^i = \sum_{i=0}^{\infty} \frac{1}{i!}\tau^i \varphi^i, \tag{4}$$

where $\varphi = (\varphi_{ij}) \in M(n, \mathbf{R}^P)$ is a fixed matrix of functions $\varphi_{ij} : P \to \mathbf{R}$, $\tau \in \mathbf{R}^P$. In the formula (4) $\tau\varphi = \tau \cdot (\varphi_{ij}) = (\tau \cdot \varphi_{ij}) \in M(n, \mathbf{R}^P)$ is a matrix of functions such that for any point $p \in P$ we have

$$(\tau \cdot \varphi_{ij})(p) = (\tau(p) \cdot \varphi_{ij}(p)) \in M(n, \mathbf{R}).$$

We can write the formula (4) in the form

$$\bar{\psi}(\tau) = \mathbf{I} + \tau\varphi + \frac{1}{2}(\tau\varphi)^2 + \frac{1}{3!}(\tau\varphi)^3 + \ldots,$$

where $\mathbf{I} = \begin{pmatrix} \mathbf{1} & & \theta \\ & \ddots & \\ \theta & & \mathbf{1} \end{pmatrix} \in \mathrm{GL}(n, \mathbf{R}^P)$ is a unitary matrix of functions, $\mathbf{1} : P \to \mathbf{R}$ and $\theta : P \to \mathbf{R}$ are the constant mappings such that $\mathbf{1}(p) = 1$ for all $p \in P$ and $\theta(p) = 0$ for all $p \in P$.

The mapping $\bar{\psi}$ defined by the formula (4) has the following properties:

1. $\bar{\psi}(\theta)(p) = I \in \mathrm{GL}(n, \mathbf{R})$ for all $p \in P$,
2. $\bar{\psi}(\tau_1 + \tau_2)(p) = \bar{\psi}(\tau_1)(p)\bar{\psi}(\tau_2)(p)$ for all $p \in P$,
3. $(\frac{d}{d\tau}\bar{\psi}(\tau) \mid_{\tau=\theta})(p) = \varphi(p)$ for all $p \in P$.

We obtain the 1-parameter subgroup of $\mathrm{GL}(n, \mathbf{R}^P)$ generated by an element $\varphi \in M(n, \mathbf{R}^P)$:

$$\bar{a}_\tau = \bar{\psi}(\tau) = \exp_{\bar{G}(P)}(\tau\varphi) \in \mathrm{GL}(n, \mathbf{R}^P).$$

The Lie algebra of linear Lie group $\mathrm{GL}(n, \mathbf{R}^P)$ is the set of all $n \times n$ matrices with elements in $\mathbf{R}^P$, $\mathcal{L}(\mathrm{GL}(n, \mathbf{R}^P)) = M(n, \mathbf{R}^P)$. The exponential mapping is

$$\exp_{\bar{G}(P)} : \mathcal{L}(\mathrm{GL}(n, \mathbf{R}^P)) \to \mathrm{GL}(n, \mathbf{R}^P),$$

$$\exp_{\bar{G}(P)}(\varphi) = \bar{a}_{\mathbf{1}} = \bar{\psi}(\mathbf{1}), \tag{5}$$

where $\mathbf{1} : P \to \mathbf{R}$ is the constant mapping such that $\mathbf{1}(p) = 1$ for all $p \in P$.
Let us note that $\bar{a}_\epsilon(p) \in \mathrm{GL}(n, \mathbf{R})$ for all $p \in P$.

Corollary 4.1 *The exponential map* $\exp_{\bar{G}(P)}$ *on the Lie algebra* $\mathcal{L}(\mathrm{GL}(n, \mathbf{R}^P))$ *can be written by the formula*

$$\exp_{\bar{G}(P)}(\varphi) = \sum_{i=0}^{\infty} \frac{1}{i!}\varphi^i, \tag{6}$$

where the above series is pointwise convergent, i.e. $(\exp_{\bar{G}(P)}(\varphi))(p) = \sum_{i=0}^{\infty} \frac{1}{i!}\varphi^i(p)$ is convergent for all $p \in P$.

5. Conclusion

To cocnclude, the functorial group $\bar{\mathrm{G}}$ can be considered as the natural generalisation of the Lie group G. The Grothendieck functor $G \mapsto \bar{\mathrm{G}}$ infers an isomorphism of the C^∞ - algebra $C^\infty(G)$ of the Lie group G and the C^∞ - algebra $C^\infty(\bar{G}(P))$ of functorial group $\bar{G}(P)$ on a stage P. The basic concepts of the Lie group G have equivalents in the functorial group $\bar{G}(P)$ at a stage P and in the functorial group $\bar{\mathrm{G}}$ at any stage. Each concept on the C^∞ - algebra $C^\infty(G)$ is associated with the corresponding concept of the C^∞ - algebra $C^\infty(\bar{G}(P))$. Despite the isomorphism of the C^∞ - algebras $C^\infty(G)$ and $C^\infty(\bar{G}(P))$, the differential groups $(G, C^\infty(G))$ and $(\bar{G}(P), C^\infty(\bar{G}(P)))$ are not isomorphic as ordered pairs, because sets of points G and $\bar{G}(P)$ are not bijective.

The study of the properties of functorial groups will be continued, owing to the substantial role of these symmetry groups and the range of their applications, particularly in the theory of functorial principal bundles associated with the corresponding classical principal bundles.

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4. Accelerated Expansion of the Universe in Terms of Differential Spaces

Jacek Gruszczak

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ARTICLE

Accelerated Expansion of the Universe in Terms of Differential Spaces

Jacek Gruszczak*

Copernicus Center for Interdisciplinary Studies, ul. Szczepańska 1/5, 31-011 Kraków, Poland
*Corresponding author. Email: sfgruszc@cyf-kr.edu.pl

Abstract
The concept of closures of manifolds in the category of Sikorski's differential spaces is applied to a description of the flat FRW models. The smoothness condition coming from this approach constitutes a strong restriction on the time dependence of the scale factor and on the energy density of the matter content of the resulting model. We demonstrate that our model agrees with the $H(z)$ dependence obtained with the help observational data concerning the type Ia supernovae, BAO and the CMB peaks tests. The model contains a string gas, two types of domain walls, four types of cosmological vacuums and a cosmological constant whose value — determined by the model — agrees with the results of the Planck Mission.

Keywords: accelerated cosmological models, differential spaces

1. Introduction

Almost at the beginning of the general relativity theory researchers noticed problems connected with the initial singularity in cosmology. There have been several attempts to overcome these problems. Below we list a few of them.

The first approach is based on the assumption that there is a possibility to construct a classical gravity theory that is free of singularities. In this strategy the potential alternative theory of gravity must contain GR as the weak field limit Einstein 1945, 1948, 1955. One of the proposals of such a theory is based on non-Riemannian geometry Cornish and Moffat 1994,Damour, Deser, and McCarthy 1993,Dobrowolski and Koc 2015 in which the metric tensor can be split into the symmetric part and the skew-symmetric part. From the mathematical point of view non-Riemannian geometry enables to circumvent the assumptions of the Hawking – Penrose singularity theorems Hawking and Penrose 1970.

The second approach involves noncommutative geometry Connes 1994; Madore 1999; Gracia-Bondía, Várilly, and Figueroa 2001; Heller, Sasin, and Lambert 1997; Heller, Pysiak, and Sasin 2005; Heller et al. 2015. This vast branch of modern mathematical physics aims at reconstructing the differential-geometrical notions *more algebraico* (most notably, in the language of Connes' spectral triples), and then, by abandoning the requirement that the algebras involved be commutative, it is believed to provide a unified mathematical framework for the study of both relativity and quanta. In doing so, the very notion of a space-time point is replaced with a more structuralized, global object, and the troublesome singularities no longer appear.

In the third approach it is presumed that all singularities disappear at the more fundamental, quantum level of the gravity theory Rosenfeld 1930b, 1930a; Rovelli and Smolin 1995a, 1995b; Markopoulou and Smolin 1998; Barrett and Crane 1998; Baez and Crane 1998, 1999. It is believed that the relation between quantum gravity and general relativity is similar to the relation between classical hydrodynamics (i.e. a theory that admits singularities) and the microscopic description of a fluid which, due to the finite size of the molecules, is free of singularities.

However, one might perceive the initial singularity not as a theoretical obstacle to be overcome, but rather as a real feature of the Universe. The main stream of research based upon this view comprises the theories of singular boundaries: g-boundaries, b-boundaries, c-boundaries, a-boundaries R. Geroch 1971, R. P. Geroch 1968, Schmidt 1971, Geroch, Kronheimer, and Penrose 1972, Geroch and Horowitz 1979, Scott and Szekeres 1994 and others.

The method approach we adopt in the present paper in the same vein as the last of the above-mentioned ones. Since the category of manifolds is a subcategory of the category Sikorski's differential spaces (called d-spaces for short) , therefore one can define closures of the flat Friedmanian model manifolds which are d-spaces Sikorski 1967, Sikorski 1971, Sikorski 1972, Waliszewski 1972, Sasin, Heller, and Multarzyński 1989, Gruszczak, Heller, and Multarzynski 1988, Gruszczak and Heller 1993. As a final result we obtain the so-called smoothness equation which we discuss in Section 2. However, this discussion will be descriptive. We do not want to burden this paper with an excessive mathematical abstraction, but rather we want to concentrate on results that may have meaning in cosmology.

A method for solving the smoothness equation is presented in Section 3. The explicit form of the solutions of the smoothness equation is shown in Section 4 (see also Gruszczak 2014). Discussion on the matter content of our model is carried out in Section 5. In Section 6 we compare our model with the observational data. The final discussion and summary are provided in Section 7.

2. On the smoothness equation

We restrict our considerations to the homogeneous, isotropic and flat cosmological models described by the FRW metric

$$g = c^2 dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{1}$$

where $a(t)$ is the scale factor, t the cosmological time. In addition, we assume that the model has an initial singularity at $t = 0$, $a(t = 0) = 0$, and starts its evolution with the velocity $\dot{a}(t = 0) > 0$.

The serious problem of the FRW cosmological models is that their manifold structures break down at $t = 0$ and therefore this moment cannot be included in the description of their time evolution. The solution we propose below is to perform the closure of the flat cosmological models in the class of differential spaces, objects more general than manifolds. This enables one to prolong the time orientability notion to the edges of the closures –called the *differential closures* (*d-closures*)– and thus to include the moment $t = 0$ into our investigations. Let us emphasize that this prolongation cannot be, in general, realized by means of the singular boundaries from the classical theory of singularities (see Heller et al. 1992). Our method was discussed in detail in Gruszczak 2014.

Including the beginning of time t in the above-described way imposes the following restrictive condition, called the *smoothness equation*

$$\dot{a}(t) = f\left(a(t), a(t)\int_0^t \frac{d\tau}{a'(\tau)}\right), \quad a(t = 0) = 0, \quad f(0,0) > 0, \tag{2}$$

where $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is a function such that the physical dimensions of the left and right sides of the smoothness equation are the same. The condition $f(0,0) > 0$ comes from the physical assumption that our Universe starts its expansion with positive "velocity".

For example, if we assume that f is an affine function of variables $f(x,y) = \beta + \gamma_1 x + \gamma_2 y$, where $(x,y) \in \mathbb{R}^2$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $\beta > 0$ then the smoothness equation (2), after rescaling, takes the form

$$\dot{\bar{a}}(t) = 1 + \gamma_1 \bar{a}(t) + \bar{\gamma}_2 \bar{a}(t)\int_0^t \frac{d\tau}{\bar{a}'(\tau)}, \quad \bar{a}(t = 0) = 0, \tag{3}$$

where $\bar{a}(t) := a(t)/\beta$ and $\bar{\gamma}_2 := \gamma_2/\beta$. Equation (3) will be called *the simplest smoothness equation*. Cosmological models (1) with scale factors satisfying the smoothness equation we shall call *the models evolving smoothly from the very beginning* or *the smoothly evolving models* or *the SE-models* for brevity.

The smoothness equation was introduced in Gruszczak 2014. On the physical side, it guarantees that for models (1) satisfying the equation the time orientability given by the vector field representing the cosmological time t can be smoothly prolonged to the moment $t = 0$. It is worth adding that in the theory of d-spaces the smoothness notion is more general than in the theory of manifolds.

3. The method of solving the simplest smoothness equation

In order to solve the smoothness equation (3), let us introduce the following auxiliary function

$$\bar{v}(t) = \int_0^t \frac{d\tau}{\bar{a}'(\tau)}. \tag{4}$$

Since in model (1) one has $\dot{\bar{a}}(t = 0) > 0$, the domain $[0, t_f]$ of $\bar{v}(t)$ is defined by the condition $\dot{\bar{a}}(t) \geqslant 0$. Depending on the values of γ_1 and $\bar{\gamma}_2$, the value of t_f is either finite or infinite. The $\bar{v}(t)$ is an increasing function in its domain and therefore it has the inverse function $t(\bar{v})$. It is worth noticing that, in a similar vein, the assumption $\dot{\bar{a}}(t) > 0$ ensures the existence of the function inverse to $\bar{a}(t)$ denoted as $t(\bar{a})$.

Now, we can rewrite the smoothness equation in the form featuring $\bar{v}(t)$

$$\dot{\bar{a}}(t) = 1 + (\gamma_1 + \bar{\gamma}_2\bar{v}(t))\bar{a}(t). \tag{5}$$

After the change of variables $\dot{\bar{a}}(t) = d\bar{a}(\bar{v})/d\bar{v}|_{\bar{v}=\bar{v}(t)} \cdot d\bar{v}(t)/dt = d\bar{a}(\bar{v})/d\bar{v}|_{\bar{v}=\bar{v}(t)} \cdot \dot{\bar{a}}(t)^{-1}$ we obtain the formula

$$\dot{\bar{a}}(t)^2 = d\bar{a}(\bar{v})/d\bar{v}|_{\bar{v}=\bar{v}(t)}$$

which enables one to write the smoothness equation in the following useful form

$$d\bar{a}(\bar{v})/d\bar{v} = (1 + \bar{a}(\bar{v})(\gamma_1 + \bar{\gamma}_2\bar{v}))^2, \quad \bar{a}(\bar{v} = 0) = 0. \tag{6}$$

This equation is solvable by elementary methods. Its solution depends on the two external parameters γ_1 and $\bar{\gamma}_2$ coming from the smoothness equation (3).

This enables us also to set down the relation between the variables t and $\bar{v}$ or, in other words, to find the function $t(\bar{v})$ inverse to the function $\bar{v}(t)$. Indeed, since $d\bar{v}/dt = 1/\dot{\bar{a}}(t)$ therefore the inverse function satisfies

$$dt(\bar{v})/d\bar{v} = \dot{\bar{a}}(t)|_{t=t(\bar{v})} =: \dot{\bar{a}}(\bar{v}). \tag{7}$$

The function $\dot{\bar{a}}(t)$ is given by the smoothness equation (3). Its value at $t = t(\bar{v})$ is $\dot{\bar{a}}(\bar{v}) = 1 + \bar{a}(\bar{v})(\gamma_1 + \bar{\gamma}_2\bar{v})$. Thus, equation (7) takes the following new form

$$dt(\bar{v})/d\bar{v} = 1 + \bar{a}(\bar{v})(\gamma_1 + \bar{\gamma}_2\bar{v}). \tag{8}$$

The scale factor $\bar{a}(\bar{v})$ is a known function. It is the solution of equation (6). Therefore, equation (8) is integrable and its solution with the initial condition $t(\bar{v} = 0) = 0$ has the form

$$t(\bar{v}) = \int_0^{\bar{v}} (1 + \bar{a}(\bar{v}')(\gamma_1 + \bar{\gamma}_2\bar{v}'))d\bar{v}'. \tag{9}$$

It is worth to notice that the pair $(\bar{a}(\bar{v}), t(\bar{v}))$ is a parametric solution of the smoothness equation (3), where $\bar{a}(\bar{v})$ satisfies equation (6) and $t(\bar{v})$ is given by formula (9).

4. Solutions of the simplest smoothness equation

Flat cosmological models with scale factors satisfying the simplest smoothness equation were studied in Gruszczak 2014. In the case $\gamma_1 < 0$ and $\gamma_2 > 0$ the SE-model turned out to exhibit an interesting evolution which is qualitatively consistent with the results of observations of type Ia supernovae. Therefore, let us restrict further considerations to SE-models with the parameters from the range $\gamma_1 \leq 0$ and $\bar{\gamma}_2 \geq 0$.

The smoothness equation written in the form (6) is of the Riccati type. Its solution with the initial condition $\bar{a}(\bar{\nu} = 0) = 0$ reads

$$\bar{a}(\bar{\nu}) = \frac{1}{\sqrt{\bar{\gamma}_2}\coth(\sqrt{\bar{\gamma}_2}\,\bar{\nu}) - \gamma_1 - \bar{\gamma}_2\bar{\nu}}, \quad \bar{\gamma}_2 > 0 \tag{10}$$

and in the case $\bar{\gamma}_2 = 0$ it is given by the formula

$$\bar{a}(\bar{\nu}) = \frac{\bar{\nu}}{1 - \gamma_1\bar{\nu}}. \tag{11}$$

Let us notice that in the case $\bar{\gamma}_2 > 0$ the final moment of evolution corresponds to $\bar{a} \to \infty$. Therefore, the final value $\bar{\nu}_f$ of the parameter $\bar{\nu}$ is the solution of the following equation

$$\sqrt{\bar{\gamma}_2}\coth(\sqrt{\bar{\gamma}_2}\,\bar{\nu}_f) - \gamma_1 - \bar{\gamma}_2\bar{\nu}_f = 0. \tag{12}$$

Thus, in this case $\bar{\nu} \in [0, \bar{\nu}_f)$.

When $\bar{\gamma}_2 = 0$ the domain of $\bar{a}(\bar{\nu})$ is the set $[0, \infty)$.

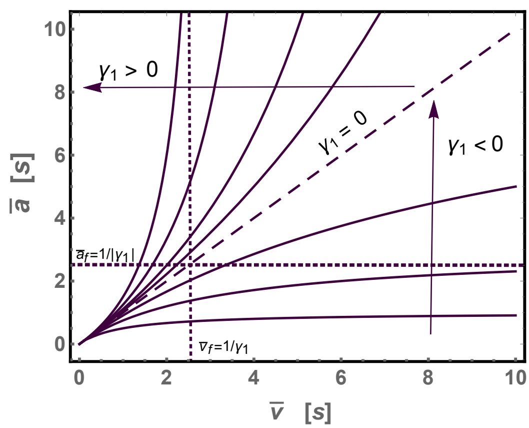

Figure 1. Scale factor $\bar{a}(\bar{\nu})$ for the SE-model with $\gamma_2 = 0$ and $\gamma_1 \in \mathbb{R}$. When $\gamma_1 = 0$ the model expands with a constant velocity. For $\gamma_1 > 0$ the SE-model accelerates from the very beginning while for $\gamma_1 < 0$ it decelerates and $\bar{a}(\bar{\nu}) \to \bar{a}_f = 1/|\gamma_1|$ when $\bar{\nu} \to \infty$. In the last case the model becomes the Minkowski space-time in the final stage of its evolution .

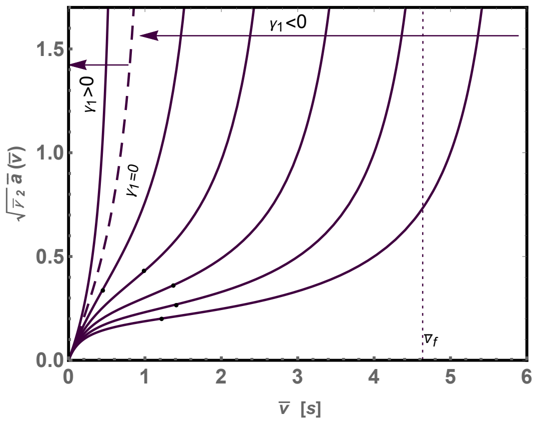

Figure 2. The re scaled scale factor $\sqrt{\gamma_2}\,\bar{a}(\bar{v})$ of the SE-model in the case $\gamma_2 > 0$ and $\gamma_1 \in \mathbb{R}$. Every curve on the plot has a vertical asymptote at $\bar{v} = \bar{v}_f$, where $\bar{v}_f$ satisfies equation (12). When $\gamma_1 \geqslant 0$ the model accelerates from the very beginning. For $\gamma_1 < 0$ the model initially decelerates but at moments indicated by small black points on the graph an accelerated evolution commences.

5. On the matter content of the SE-model

In the standard cosmology one employs the following methodological scenario. One assumes what kind(s) of fluid(s) permeate(s) the cosmological model and then, on this basis, deduces the evolutionary properties of the cosmological model solving Friedman's equations. In the case of the flat models, which we shall discuss in the current paper, the equations read

$$\tilde{\varepsilon}(t) := \frac{3}{\kappa c^2}\frac{\dot{a}(t)^2}{a(t)^2}, \qquad \tilde{p}(t) := -\frac{1}{\kappa c^2}\left(2\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}(t)^2}{a(t)^2}\right), \tag{13}$$

$$\tilde{\varepsilon} = \varepsilon + \Lambda/\kappa, \qquad \tilde{p} = p - \Lambda/\kappa, \tag{14}$$

where $\kappa = 8\pi G/c^4$, Λ is the cosmological constant and ε and p denote the energy density and pressure of all assumed kinds of fluids filling up the model investigated. The form of $\tilde{\varepsilon}$ and $\tilde{p}$ in formula (14) depends on our choice. We can work with or without the cosmological constant Λ. If we decide to work with a nonzero cosmological constant then the Λ appears in the solutions of equations (13) as an additional parameter.

In the present paper we reverse the standard methodological scenario outlined above. We assume that our model evolves from the initial singularity according to solutions (10) or (11). On this basis we try to reconstruct the matter content of the SE-model with the help of equations (13). Now $\tilde{\varepsilon}$ and $\tilde{p}$ are known functions defined by the right-hand sides of equations (13). They do not depend on any additional parameters, in particular on Λ. Therefore, in the present context, we cannot simply choose Λ to be zero or non-zero. However, we can still employ physical argumentation. Concretely, it is reasonable to assume that all forms of the usual matter should 'disperse' as the Universe expands to infinite size. In that case, the energy density ε and pressure p of every kind of fluid should vanish. This requirement can be expressed as follows: $\lim_{a\to\infty}\varepsilon(a) = 0$, $\lim_{a\to\infty}p(a) = 0$. Therefore, if only limits of our $\tilde{\varepsilon}(a)$ and $\tilde{p}(a)$ satisfy the following condition

$$\lim_{a\to\infty}\tilde{\varepsilon}(a) = -\lim_{a\to\infty}\tilde{p}(a) \neq 0 \tag{15}$$

we can define, with the help of formulas (14), the cosmological constant Λ_{th} in our model

$$\Lambda_{\text{th}} := \kappa \lim_{a\to\infty}\tilde{\varepsilon}(a). \tag{16}$$

Let us return to the discussion of the SE-model. First we notice that the Hubble function can be obtained with the help of the smoothness equation (3)

$$H(t) := \dot{\bar{a}}(t)/\bar{a}(t) = 1/\bar{a}(t) + \gamma_1 + \bar{\gamma}_2\bar{v}(t). \tag{17}$$

Then equations (13) for solutions of the smoothness equation can be rewritten in the form

$$\tilde{\varepsilon}(t) = 3H(t)^2/\kappa c^2, \tag{18}$$

$$\tilde{p}(t) = \frac{2}{\kappa c^2\bar{a}(t)}[H(t) - \bar{\gamma}_2/H(t)] - 3H(t)^2/\kappa c^2. \tag{19}$$

Since the solutions of the smoothness equation are known functions, the right-hand sides of formulas (18) and (19) are treated now as the definitions of $\tilde{\varepsilon}(t)$ and $\tilde{p}(t)$ respectively.

Now, $H(t)$ can be expressed in the form dependent on $\bar{a}$

$$H(\bar{a}) := H(t)|_{t=t(\bar{a})} = 1/\bar{a} + \gamma_1 + \bar{\gamma}_2\bar{v}(\bar{a}), \tag{20}$$

where $\bar{v}(\bar{a}) := \bar{v}(t)|_{t=t(\bar{a})}$ is the inverse function of the known scale factor $\bar{a}(\bar{v})$ (see formulas (10) and (11)). This form of H is suitable for the following discussion.

Thus, also the $\tilde{\varepsilon}$ in our model can be presented in the form depended on the variable $\bar{a}$

$$\tilde{\varepsilon}(\bar{a}) := 3H(\bar{a})^2/\kappa c^2 = \left[\frac{3}{\bar{a}^2} + \frac{6\gamma_1}{\bar{a}} + 3\gamma_1^2 + 3\bar{\gamma}_2^2\bar{v}(\bar{a})^2 + \frac{6\bar{\gamma}_2\bar{v}(\bar{a})}{\bar{a}} + 6\gamma_1\bar{\gamma}_2\bar{v}(\bar{a})\right]/\kappa c^2. \tag{21}$$

This expression, however, not yet ready redy for interpretation because the function $\bar{v}(\bar{a})$ is not given in an explicit form. Nevertheless, one can extract certain properties of $\bar{v}(\bar{a})$ in the neighbourhood of $\bar{a} = 0$ by means of the series expansion of $\bar{a}(\bar{v})$ at $\bar{v} = 0$. The first terms of the inverse series read

$$\bar{v}(\bar{a}) = \bar{a} - \gamma_1\bar{a}^2 + \frac{1}{3}(3\gamma_1^2 - 2\bar{\gamma}_2)\bar{a}^3 + \ldots \quad . \tag{22}$$

This means that $\bar{v}(\bar{a})$ can be written as

$$\bar{v}(\bar{a}) = \bar{a}(1 + \psi(\bar{a})),$$

where the map ψ satisfies the condition $\psi(\bar{a} = 0) = 0$. Thanks to this observation we can set down the correct form of the vacuum term in formula (21)

$$\tilde{\varepsilon}(\bar{a}) = \left[\frac{3}{\bar{a}^2} + \frac{6\gamma_1}{\bar{a}} + 3(\gamma_1^2 + 2\bar{\gamma}_2) + 3\bar{\gamma}_2^2\bar{v}(\bar{a})^2 + 6\bar{\gamma}_2\psi(\bar{a}) + 6\gamma_1\bar{\gamma}_2\bar{v}(\bar{a})\right]/\kappa c^2. \tag{23}$$

To recognize fluids that fill up the SE-model we apply the traditional interpretation method which depends on the form of $\tilde{\varepsilon}(\bar{a})$ and the barotropic index w usually used in cosmology. Necessary formulas for $\tilde{p}(\bar{a})$ can be obtained with the help of equality (19) and the composition $\tilde{p}(\bar{a}) = \tilde{p}(t)|_{t=t(\bar{a})}$.

In order to guess the matter content of the SE-model with the late acceleration, $(\gamma_1 < 0, \bar{\gamma}_2 > 0)$, let us first discuss its behavior for γ-parameters satisfying: $(\gamma_1 = 0, \bar{\gamma}_2 = 0)$, $(\gamma_1 < 0, \bar{\gamma}_2 = 0)$ and $(\gamma_1 = 0, \bar{\gamma}_2 > 0)$.

Example 1 *The SE-model with* $\gamma_1 = 0$ *and* $\bar{\gamma}_2 = 0$.

In this case formulas (19–20) yield

$$\tilde{\varepsilon}(\bar{a}) = 3/\kappa c^2\bar{a}^2, \quad \tilde{p}(\bar{a}) = -1/\kappa c^2\bar{a}^2. \tag{24}$$

In this model there are no reasons to introduce the cosmological constant and therefore $\tilde{\varepsilon} = \varepsilon$ and $\tilde{p} = p$. Thus, the model is filled with a string gas with the equation of state $p/\varepsilon = -1/3$. The string gas causes an unlimited expansion of such a universe with the constant velocity $\dot{\bar{a}} = 1$ (see Figure (1)). It is worth to notice that cosmological models with a nonzero curvature exhibit a similar dependence ε of a ($\varepsilon \propto 3/a^2$). One can say that the string gas substitutes the curvature in flat cosmological models Dąbrowski and J. 1989; Dąbrowski 1996; Kamenshchik and Khalatnikov.

Example 2 *The SE-model with* $\gamma_1 < 0$ *and* $\bar{\gamma}_2 = 0$.

In this case formulas on effective energy density and pressure have the form

$$\tilde{\varepsilon}(\bar{a}) = \left(\frac{3}{\bar{a}^2} + \frac{6\gamma_1}{\bar{a}} + 3\gamma_1^2\right)/\kappa c^2, \tag{25}$$

$$\tilde{p}(\bar{a}) = \left(-\frac{1}{\bar{a}^2} - \frac{4\gamma_1}{\bar{a}} - 3\gamma_1^2\right)/\kappa c^2. \tag{26}$$

From the viewpoint of the traditional interpretation our model is filled with three types of fluids: a string gas, domain walls and a cosmological vacuum. These class of models were considered in papers Dąbrowski 1996; Dąbrowski and Larsen 1995. In what follows we shall call the domain walls and the vacuum the γ_1-*domain walls* and the γ_1-*vacuum* respectively. Let us notice that one can interpret the γ_1-domain walls term as a potential term since $\gamma_1 < 0$.

Let us look once more on the evolutionary properties of the SE-model shown in Figure 1. The scale factor $\bar{a}$ is an increasing function of the variable $\bar{v}$ and $\lim_{\bar{v}\to\infty} \bar{a}(\bar{v}) = 1/|\gamma_1| =: \bar{a}_f$. It means geometrically that such a universe asymptotically becomes the Minkowski space-time for every $\gamma_1 < 0$. This fact is mirrored in the behaviour of $\tilde{\varepsilon}(\bar{a})$ at $\bar{a}_f$

$$\lim_{\bar{a}\to\bar{a}_f} \tilde{\varepsilon}(\bar{a}) = \left(3\gamma_1^2 + 6\gamma_1|\gamma_1| + 3\gamma_1^2\right)/\kappa c^2 = 0^+. \tag{27}$$

It means that the γ_1-vacuum is not a passive vacuum but rather that it interacts with the string gas and the γ_1-domain walls causing that the final energy density and the final pressure to be zero. There are no reasons to introduce the cosmological constant in this model.

Example 3 *The SE-model with* $\gamma_1 = 0$ *and* $\bar{\gamma}_2 > 0$.

Now formulas on $\tilde{\varepsilon}(\bar{a})$ and $\tilde{p}(\bar{a})$ look as follows

$$\tilde{\varepsilon}(\bar{a}) = \left[\frac{3}{\bar{a}^2} + 6\bar{\gamma}_2 + 3\bar{\gamma}_2^2\bar{v}(\bar{a})^2 + 6\bar{\gamma}_2\psi(\bar{a})\right]/\kappa c^2, \tag{28}$$

$$\tilde{p}(\bar{a}) = \left[-1/\bar{a}^2 - 6\bar{\gamma}_2 - 3\bar{\gamma}_2^2\bar{v}(\bar{a})^2 - 4\bar{\gamma}_2\psi(\bar{a}) + 2\bar{\gamma}_2^2\bar{a}\bar{v}(\bar{a})/(1 + \bar{\gamma}_2\bar{a}\bar{v}(\bar{a}))\right]/\kappa c^2. \tag{29}$$

Analyzing subsequent terms in (28) one can see that the our model contains the string gas, a cosmological vacuum and an unknown fluid represented by the term $3\bar{\gamma}_2^2\bar{v}(\bar{a})^2/\kappa c^2$. The last term one can interpret as a potential energy density since $6\bar{\gamma}_2\psi(\bar{a})/\kappa c^2 \leq 0$.

An additional interpretation is provided by the comparison of $\tilde{\varepsilon}$ with $\tilde{p}$. The barotropic indexes w of the four terms in $\tilde{\varepsilon}$ and $\tilde{p}$ have the following values –1/3, –1, –1 and –2/3, respectively. It is a surprising fact that the unknown fluid represented by the term $3\bar{\gamma}_2^2\bar{v}(\bar{a})^2/\kappa c^2$ can be linked with a vacuum and that the potential energy density term $6\bar{\gamma}_2\psi(\bar{a})/\kappa c^2$ can be associated with domain walls. The last term in $\tilde{p}(\bar{a})$, which is a perturbation of Dalton's law, is a suggestion that there is an interaction between (some of) the fluids contained in the SE-model.

The fluids represented by the terms $3/\kappa c^2\bar{a}^2$, $6\bar{\gamma}_2/\kappa c^2$, $3\bar{\gamma}_2^2\bar{v}(\bar{a})^2/\kappa c^2$, $6\bar{\gamma}_2\psi(\bar{a})/\kappa c^2$ appearing in formula (29) will be called the *string gas*, the γ_2-*vacuum*, the γ_2^b-*vacuum* and the γ_2^b-*domain walls*, respectively. The superscript "b" refers to the fact that the fluid in question is defined by means of the barotropic index.

More information on our SE-model is provided by the asymptotic behaviour of the terms in the formulas for $\tilde{\varepsilon}(\bar{a})$ and $\tilde{p}(\bar{a})$

$$\tilde{\varepsilon}^f := \lim_{\bar{a}\to\infty} \tilde{\varepsilon}(\bar{a}) = \left(0 + \underline{6\bar{\gamma}_2} + 3\bar{\gamma}_2^2\bar{v}_f^2 - \underline{6\bar{\gamma}_2}\right)/\kappa c^2 = 3\bar{\gamma}_2^2\bar{v}_f^2/\kappa c^2, \tag{30}$$

$$\tilde{p}^f := \lim_{\bar{a}\to\infty} \tilde{p}(\bar{a}) = \left(0 - \underline{6\bar{\gamma}_2} - 3\bar{\gamma}_2^2\bar{v}_f^2 + \underline{4\bar{\gamma}_2} + \underline{2\bar{\gamma}_2}\right)/\kappa c^2 = -3\bar{\gamma}_2^2\bar{v}_f^2/\kappa c^2, \tag{31}$$

where $\tilde{\varepsilon}^f$ and $\tilde{p}^f$ denote the effective final energy density and the effective final pressure of matter. In formula (30) the γ_2-vacuum term is canceled by the γ_2^b-domain walls term while in formula (31) the γ_2-vacuum term is canceled by the γ_2^b-domain walls term and the term perturbing the Dalton's law. It indicates that between the γ_2-vacuum and the γ_2^b-domain walls there is an interaction. We have no suggestions that the string gas and the γ_2^b-vacuum are interacting fluids in the discussed mixture.

Let us notice that $\tilde{\varepsilon}^f$ and $\tilde{p}^f$ satisfy the following inequality $\tilde{\varepsilon}^f = -\tilde{p}^f = 3\bar{\gamma}_2^2\bar{v}_f^2/\kappa c^2 \neq 0$. Therefore we can introduce the cosmological constant $\Lambda_{\text{th}} = \kappa\tilde{\varepsilon}^f = 3\bar{\gamma}_2^2\bar{v}_f^2/c^2$ to our model. Plugging the above formulas for $\tilde{\varepsilon}$, $\tilde{p}$ and Λ_{th} into (14), we obtain

$$\varepsilon(\bar{a}) = \left[\frac{3}{\bar{a}^2} + 6\bar{\gamma}_2 + 3\bar{\gamma}_2^2(\bar{v}(\bar{a})^2 - \bar{v}_f^2) + 6\bar{\gamma}_2\psi(\bar{a})\right]/\kappa c^2, \tag{32}$$

$$p(\bar{a}) = \left[-1/\bar{a}^2 - 6\bar{\gamma}_2 - 3\bar{\gamma}_2^2(\bar{v}(\bar{a})^2 - \bar{v}_f^2) - 4\bar{\gamma}_2\psi(\bar{a}) + 2\bar{\gamma}_2^2\bar{a}\bar{v}(\bar{a})/(1 + \bar{\gamma}_2\bar{a}\bar{v}(\bar{a}))\right]/\kappa c^2. \tag{33}$$

Evidently, $\lim_{\bar{a}\to\infty}\varepsilon(\bar{a}) = \lim_{\bar{a}\to\infty} p(\bar{a}) = 0$. But the analogous limit for the barotropic index

$$\lim_{\bar{a}\to\infty} w(\bar{a}) = \lim_{\bar{a}\to\infty}\frac{p(\bar{a})}{\varepsilon(\bar{a})} = \lim_{\bar{v}\to\bar{v}_f}\frac{p(\bar{v})}{\varepsilon(\bar{v})} = -2/3$$

leads to an interesting conclusion that the final evolution stage of our model is dominated by the γ_2^b-domain walls.

The discussed SE-model filled with the string gas and the γ_2-fluids is subject to the accelerated expansion to infinity from the very beginning (see Figure 2).

Now, we are ready to discuss the matter content of the SE-model with the late acceleration (see Figure (2)).

Example 4 *The SE-model with* $\gamma_1 < 0$ *and* $\bar{\gamma}_2 > 0$.

Almost all properties of the previous examples are now cumulated. Formulas on $\tilde{\varepsilon}(\bar{a})$ and $\tilde{p}(\bar{a})$ have now the more complicated form

$$\tilde{\varepsilon}(\bar{a}) = \left(\frac{3}{\bar{a}^2} + \frac{6\gamma_1}{\bar{a}} + 3\gamma_1^2 + 6\gamma_1\bar{\gamma}_2\bar{v}(\bar{a}) + 3\bar{\gamma}_2^2\bar{v}(\bar{a})^2 + 6\bar{\gamma}_2 + 6\bar{\gamma}_2\psi(\bar{a})\right)/\kappa c^2, \tag{34}$$

$$\begin{aligned}\tilde{p}(\bar{a}) &= \left(-\frac{1}{\bar{a}^2} - \frac{4\gamma_1}{\bar{a}} - 3\gamma_1^2 - 6\gamma_1\bar{\gamma}_2\bar{v}(\bar{a}) - 3\bar{\gamma}_2^2\bar{v}(\bar{a})^2 - 6\bar{\gamma}_2 - 4\bar{\gamma}_2\psi(\bar{a})\right)/\kappa c^2 \\ &+ \frac{2}{\kappa c^2}\cdot\frac{\bar{\gamma}_2(\gamma_1 + \bar{\gamma}_2\bar{v}(\bar{a}))\bar{a}}{1 + (\gamma_1 + \bar{\gamma}_2\bar{v}(\bar{a}))\bar{a}}.\end{aligned} \tag{35}$$

Applying the traditional interpretation we can recognize four terms in formula (34), namely $3/\kappa c^2\bar{a}^2$, $6\gamma_1/\kappa c^2\bar{a}$, $3\gamma_1^2/\kappa c^2$ and $6\bar{\gamma}_2/\kappa c^2$. We can link these terms with the following fluids: the string gas, the γ_1-domain walls, the γ_1-vacuum and the γ_2-vacuum, respectively. The remaining terms can be interpreted with the help of the barotropic index w. The terms $-6\gamma_1\bar{\gamma}_2\bar{v}(\bar{a})/\kappa c^2$ and $3\bar{\gamma}_2^2\bar{v}(\bar{a})^2/\kappa c^2$ can be linked with cosmological vacuums which will be called the $\gamma_{1,2}^b$-*vacuum* and the γ_2^b-*vacuum*, respectively. The last undiscussed term we shall link with the γ_2^b-domain walls (see Example 3).

One can obtain some further properties of the fluids in question by considering the following limits

$$\begin{aligned}\tilde{\varepsilon}^f := \lim_{\bar{a}\to\infty} \tilde{\varepsilon}(\bar{a}) &= \left(0 - 0 + 3\gamma_1^2 + 6\gamma_1\bar{\gamma}_2\bar{v}_f + 3\bar{\gamma}_2^2\bar{v}_f^2 + \underline{6\bar{\gamma}_2} - \underline{6\bar{\gamma}_2}\right)/\kappa c^2 \\ &= 3(\gamma_1 + \bar{\gamma}_2\bar{v}_f)^2/\kappa c^2,\end{aligned} \tag{36}$$

$$\begin{aligned}\tilde{p}^f := \lim_{\bar{a}\to\infty} \tilde{p}(\bar{a}) &= \left(0 + 0 - 3\gamma_1^2 - 6\gamma_1\bar{\gamma}_2\bar{v}_f - 3\bar{\gamma}_2^2\bar{v}_f^2 - \underline{6\bar{\gamma}_2} + \underline{4\bar{\gamma}_2} + \underline{2\bar{\gamma}_2}\right)/\kappa c^2 \\ &= -3(\gamma_1 + \bar{\gamma}_2\bar{v}_f)^2/\kappa c^2.\end{aligned} \tag{37}$$

The underlined terms suggest that there is an interaction between the γ_2-vacuum and the γ_2^b-domain walls like in Example 3. The remaining terms in formulas (36) and (37) define nonzero $\tilde{\varepsilon}^f$ and $\tilde{p}^f$ such that $\tilde{p}^f = -\tilde{\varepsilon}^f$. If we assume that Λ (see formula (14)) is zero then the final energy density of all matter contained in the universe $\varepsilon^f \equiv \tilde{\varepsilon}^f > 0$. It means that in this case the universe explodes to infinity.

On the other hand, if we assume that $\Lambda \neq 0$ (see Example 3) then we can define the cosmological constant

$$\Lambda_{\text{th}} := \tilde{\varepsilon}^f = 3(\gamma_1 + \bar{\gamma}_2\bar{v}_f)^2/c^2. \tag{38}$$

The discussed SE-model with the cosmological constant $\Lambda = \Lambda_{\text{th}}$ will be called the Λ*SE-model*. For the ΛSE-model formulas on the energy density ε and pressure p have the form

$$\varepsilon(\bar{a}) = \left[\frac{3}{\bar{a}^2} + \frac{6\gamma_1}{\bar{a}} + V(\bar{a}) + 6\bar{\gamma}_2 + 6\bar{\gamma}_2\psi(\bar{a})\right]/\kappa c^2, \tag{39}$$

$$p(\bar{a}) = \left[-\frac{1}{\bar{a}^2} - \frac{4\gamma_1}{\bar{a}} - V(\bar{a}) - 6\bar{\gamma}_2 - 4\bar{\gamma}_2\psi(\bar{a}) + \frac{2\bar{\gamma}_2(\gamma_1 + \bar{\gamma}_2\bar{v}(\bar{a}))\bar{a}}{1 + (\gamma_1 + \bar{\gamma}_2\bar{v}(\bar{a}))\bar{a}}\right]/\kappa c^2, \tag{40}$$

where

$$V(\bar{a}) := 3(\gamma_1 + \bar{\gamma}_2\bar{v}(\bar{a}))^2 - 3(\gamma_1 + \bar{\gamma}_2\bar{v}_f)^2. \tag{41}$$

The term $V/\kappa c^2$ is a w-vacuum term. The fluid associated with this term shall be called the V^b-*vacuum*. It is a mixture of the γ_1-vacuum, the $\gamma_{1,2}^b$-vacuum and γ_2^b-vacuum. Since $V(\bar{a}) \leq 0$ for $\bar{a} \in [0, \infty)$, the $V(\bar{a})/\kappa c^2$ can be interpreted as a potential term in the energetic balance ε. Similarly to the result in Example 3

$$\lim_{\bar{a}\to\infty} w(\bar{a}) = \lim_{\bar{v}\to\bar{v}_f} \frac{p(\bar{v})}{\varepsilon(\bar{v})} = -2/3.$$

It means that the final evolution stage of the discussed model is dominated by the γ_2^b-domain walls.

6. Observational $H_{\rm obs}(z)$ data and the ΛSE-model

In this section we present an observational motivation for the choice of the γ-parameters from the range $\gamma_1 < 0$ and $\bar{\gamma}_2 > 0$. First of all let us notice that in our ΛSE-model the dependence $H(z)$ can be expressed in a parametric form dependent on $\bar{v}$ because both the Hubble function $H(t)$ and the redshift $z(t) := \bar{a}(t_0)/\bar{a}(t) - 1$ can be written as functions of $\bar{v}$

$$H(\bar{v}) := H(t)|_{t=t(\bar{v})} = \sqrt{\bar{\gamma}_2}\coth(\sqrt{\bar{\gamma}_2}\bar{v}), \tag{42}$$

$$z(\bar{v}) := z(t)|_{t=t(\bar{v})} = \bar{a}(\bar{v}_0)/\bar{a}(\bar{v}) - 1, \tag{43}$$

where $H(t)$ is given by formula (17) and $\bar{a}(\bar{v})$ is a known function (10). The value of $\bar{v}$ at the present moment

$$\bar{v}_0 = \operatorname{arccoth}(H_0/\sqrt{\bar{\gamma}_2})/\sqrt{\bar{\gamma}_2} \tag{44}$$

we can calculate with the help of the equality $H_0 = H(\bar{v}_0)$. The numerical value of $H_0 = 67.3\mathrm{km}/(\mathrm{s\,Mpc})$ is taken from the results of the Planck Mission *Planck Collaboration: Planck 2013 results. XVI. Cosmological parameters.*

Thus, the parametric equations for $H(z)$ can be written as

$$H(\bar{v}) = \sqrt{\bar{\gamma}_2}\coth(\sqrt{\bar{\gamma}_2}\bar{v}) \quad z(\bar{v}) = \bar{a}(\bar{v}_0)/\bar{a}(\bar{v}) - 1, \tag{45}$$

where $\bar{v} \in [0, \bar{v}_f)$ (see equation (12)).

Now, we can apply the χ^2 procedure in order to find the values of γ_1 and $\bar{\gamma}_2$ which best fit the SE-model to the recently updated observational data Zhang, Ma, and Lan 2010; Ma Cong 2011; Yu et al. 2011; Blake et al. 2012; Chuang and Wang 2013; Busca et al. 2013; Jimenez, Simon, and Verde 2005; Zhang et al. 2014; Moresco et al. 2012. As an outcome we obtain $\gamma_1 = -2.69921 \times 10^{-18}\ \mathrm{s}^{-1}$ and $\bar{\gamma}_2 = 3.1211 \times 10^{-36}\ \mathrm{s}^{-2}$ at the level of $\chi^2_{min} = 17.454$ (see Figure 3). For comparison, we used the result of the χ^2 procedure for the ΛCDM-model with the same data. The fit for the ΛCDM-model is represented by the dashed line while the prediction of the smoothly evolving model is represented by the solid line. The value of χ^2_{min} in this case is similar. Concretely, $\chi^2_{min} = 18.119$. The time

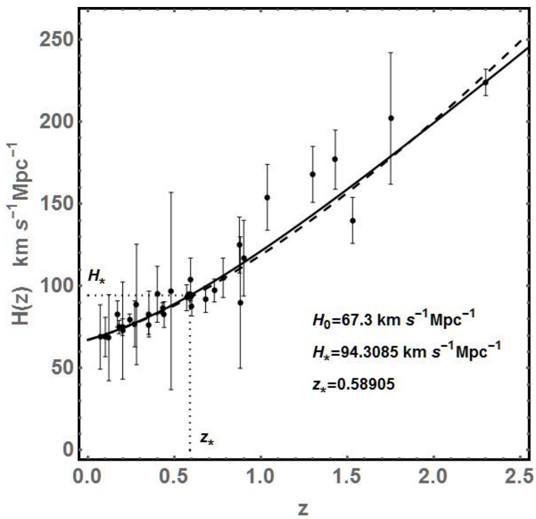

Figure 3. The best fit of the theoretical $H(z)$ dependence to the observational data of Ia type supernovae for the discussed SE-model.

variable t is a one-to-one and increasing function of $\bar{v}$ (see (9)). Therefore, every moment of the discussed time evolution can be ascribed a unique value of the $\bar{v}$ variable. The most interesting moments of the time evolution seem to be the present moment t_0 (or $\bar{v}_0$), the moment t_* (or $\bar{v}_*$) of the commencement of the late acceleration and the final moment $t \to \infty$ (or $\bar{v}_f$).

Using the obtained best-fit values of γ_1 and $\bar{\gamma}_2$ and with the help of formulas (12,38,44) and the analysis of the $\bar{a}(\bar{v})$ dependence, we can calculate $\bar{v}_0 = 6.38094 \times 10^{17}\, s$, $\bar{v}_f = 1.43795 \times 10^{18}\, s$, $\bar{v}_* = 3.73357 \times 10^{17}\, s$ and $\Lambda_{th} = 1.06803 \times 10^{-52}\,\mathrm{m}^{-2}$. These parameters enable us to obtain values of the following important quantities characteristic for our model i.e. the age of the universe $t_0 = 14.8063 \times 10^9$ y, the acceleration commencement moment $t_* = 8.96978 \times 10^9$ y, the redshift $z_* = z(\bar{v}_*) = 0.589055$ and the value of the Hubble function at the acceleration commencement moment $H_* = H(\bar{v}_*) = 94.3085 \times \mathrm{km\,s^{-1}Mpc^{-1}}$. The theoretically calculated quantities t_0, t_*, z_*, H_* and $\Lambda := \Lambda_{th}$ are thus in agreement with the observational results.

The fact that the value of Λ_{th} agrees with the results of the Planck Mission favors our ΛSE-model. For this model we can draw a graph of the $w_\Lambda(\bar{v})$ dependence (Figure 4). The Figure confirms results of Example 4 and additionally demonstrates that in the deceleration-acceleration period pressure p of matter in such universe was positive. In the next parts of the paper we will concentrate our discussion on the ΛSE-model.

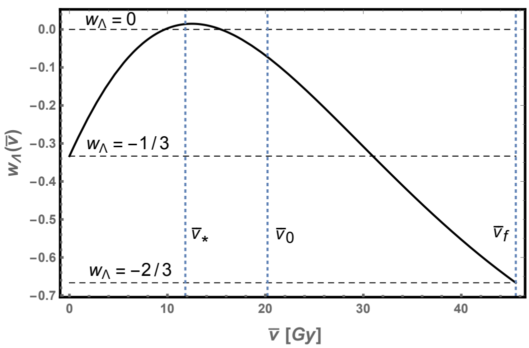

Figure 4. The $w_\Lambda(\bar{v})$ dependence for the ΛSE-model best fit to the observational data.

7. Summary

The smoothness equation (3) is a result of a strictly geometrical discussion of the assumption that orientation with respect to the cosmological time makes sense also on the manifold's d-closures for the flat FRW-models Gruszczak 2014. That assumption is a very restrictive condition. The ΛSE-model constitutes the solution of the smoothness equation. It is a very intriguing fact that this strictly geometrical considerations lead to the ΛSE-model which agrees with the observational data (see Section 6).

Our model does not contain fluids usually considered as "ordinary matter". It contains the string gas, two types of domain walls and vacuums: the γ_1-vacuum, the γ_2-vacuum, the γ_2^b-vacuum and the γ_{12}^b-vacuum. The fluids are interacting fluids.

Let us trace the role of fluids discussed in Example 4 in the important moments of the ΛSE-model evolution.

In the first stages of the evolution ($\bar{a} \approx 0$) the dominating forms of matter are the string gas and the γ_1-domain walls. The cause that our ΛSE-model decelerates expansion are the γ_1-domain walls (see Example 2). The remaining fluids have now a little influence on the rate of the expansion.

In the middle stages of the evolution ($\bar{v} \approx \bar{v}_*$ or $\hat{t} \approx \hat{t}_*$) the moderate influence of the γ_1-domain walls vanishes. Now the dominating role is played by all of the γ_2-fluids and the γ_1-vacuum. The fluids cause the change from deceleration to acceleration (Figures 2 and 4).

At the final stage of the ΛSE-universe evolution, its expansion accelerates to infinity (see Figure 2) and the dominating form of matter are the γ_2^b-domain walls (Examples 3 and 4). In the end the γ_2^b-domain walls disappear due to expansion.

In our next papers we will discuss a SE-model which additionally contains dust and radiation.

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5. Negation of the Tsirelson's conjecture and genericity in quantum mechanics

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ARTICLE

Negation of the Tsirelson's conjecture and genericity in quantum mechanics

Jerzy Król[*†] and Torsten Asselmeyer-Maluga[‡]

†University of Information Technology and Management, Chair of Cognitive Science and Mathematical Modelling, ul. Sucharskiego 2, 35-22 Rzeszów, Poland
‡German Aerospace Center (DLR), Berlin, Germany
*Corresponding author. Email: iriking@wp.pl

Abstract

We show that the negation of the Tsirelson's conjecture can be understood as, and in fact follows from, the ZFC - genericity of infinite sequences of the QM outcomes. We extend the result of Landsman that there holds the equivalence of the statistics of the entire sequences with the statistic of the single QM measurements in such sequences, over stratified infinite tensor products of the finite dimensional spaces. The stratification is based on uncomputable Turing classes and also on the Solovay genericity and the equivalence is now corrected by the appearance of certain self-adjoint operator. From studies on algorithmic randomness follows that these two properties are in a sense orthogonal one to the other for random sequences. We show two models of ZFC which indeed separate them. This is based on the classic results for Cohen and random forcings in set theory and also on Takeuti's early results on Boolean-valued models and quantum mechanics. This separation is the tool for showing that not all correlations between commuting operators on infinite dimensional Hilbert spaces can be reproduced on a tensor product of two spaces with the corresponding self-adjoint operators. We discuss the findings also from the perspective of eventual practical applications. The supplementary subsections contain discussion of QM with the set theory component along with Solovay randomness of the outcomes.

Keywords: Tsirelson's conjecture, Boolean-valued models ZFC, quantum mechanics

1. Introduction

This paper concerns the formal relationship obtaining between three research domains: one in quantum mechanics (Tsirelson's conjecture), the second in formal set theory (the existence of generic sets for the models of set theory), and the third pertaining to the algorithmic randomness of binary infinite sequences (Turing classes of uncomputability and randomness). The results, though concerned with exploring the formal side of QM and so themselves being formal in character, may hopefully contribute to a better understanding of the future possibilities for new experimental research directions. QM on infinite dimensional Hilbert spaces, and the spaces of infinite sequences of QM outputs, belong to a particular domain open for eventual extensions, e.g. ref. Coladangelo and Stark 2020. Needless to say, reconciliation with general relativity (GR) constitutes a particular task, in which such extensions may find a place, and the recent proof negating Tsirelson's conjecture furnishes a motivation for looking at infinite constructions in QM more carefully. The infinite constructions in quantum physics are rather typical and belong to the heart of quantum field theory (QFT), statistical physics, and also quantum mechanics. They usually take the form of infinite, even uncountable, tensor products of Hilbert spaces (ITP) or operators, as in the case of QFT at any point of space-time where quantum fields are defined one has, formally, infinitely many interacting quantum systems. Another example is a thermodynamical limit, where what results are infinitely many interacting quantum particles or modes, and where the case also refers to ITPs. Infinite spin chains are subsets of ITPs, while Waveguide and Resonator Quantum Electrodynamics, e.g. ref. Heuck, Jacobs, and Englund 2020, also refers to ITPs when describing infinitely many transmission lines and modes in cavities. This paper indicates that there is a gap to be investigated when one extends the standard QM formalism over infinite constructions on the one hand, and considers quantum field theory, where these infinities are just there, on the other. This gap can be regarded as furnishing a reason for the incompatibility of QM and GR that so far remains unvanquished.

Tsirelson's problem is a conjecture to the effect that the sets of finitely many correlations of independent measurements of commuting observables on a Hilbert space $\mathcal{H}$ of a quantum system will always be reproducible by the complemented set of all finitely many independent measurements on the joint system with the product Hilbert space $\mathcal{H}_a \times \mathcal{H}_b$. (See the more precise description in the Key Terminologies and Results sections.) This has been recently shown to be false Zhengfeng Ji et al. 2022: i.e. the product case does not reproduce all correlations on the entire, necessarily infinite-dimensional, Hilbert space. Tsirelson's conjecture is known to be equivalent to the Connes embedding problem for operator algebras, already stated in 1976 by Alain Connes (Connes 1976; N. Ozawa 2013). Thus resolving Tsirelson's conjecture in the negative serves to resolve the Connes embedding problem.

The genericity problem in formal set theory is the question of the existence of generic filters G in models of Zermelo Fraenkel set theory with, eventually, the axiom of choice. For first-order theories with countable languages there will always exist countable models that, in the context of set theory, guarantee that generic filters do exist. This, however, means that the countable transitive model M is nontrivially extended into the forcing extension model M[G]. There are two basic forcing procedures in set theory, which are the random and Cohen forcings, and this last, around about 1963, led Peter Cohen (the inventor of the forcing procedure in mathematics) to independence proofs of the continuum hypothesis and axiom of choice from the Zermelo-Fraenkel axioms. Since then, various forcings have been construed over the years, leading to a tremendous richness in respect of independence results and set theory constructions. It has also been proved that generic filters do not exist for the universe (cumulative hierarchy) of sets V.

In algorithmic randomness theory, various classes of Turing uncomputability and notions of the randomness of infinite binary sequences are investigated in depth, and a subtle relation between random, and non-random though uncomputable, binary sequences has been worked out. In particular, some random sequences arise as being based on random genericity reduced to arithmetic, while some others do so as being based on Cohen genericity also reduced to arithmetic, and certain limiting rules (exclusion laws) hold between the two kinds, e.g. ref. Downey and Hirschfeldt 2010.

Tsirelson's problem in QM, and the genericity problem in set theory, even if they seem to be entirely separate, are in fact tightly connected in QM, and bridged by the direct generalization of the above limiting rules for the sequences. The way in which this comes about is the main concern of this work.

Even though the methods applied resemble, to some degree, the methods in the original proof of the negation of Tsirelson's conjecture ($\neg$TC) in ref. Zhengfeng Ji et al. 2022, the focus here is rather on understanding which infinite binary sequences, under general suppositions, might distinguish TC from $\neg$TC, rather than analysing the uncomputability and complexity of the entire sets of the correlations.

The objective of this work might seem technical, or even not directly related to quantum physics. However, we think the contrary is true, especially when one considers certain unsolved central problems of physics such as the reconciliation of quantum mechanics with general relativity, or some others that could become quite important in the not-so-distant future. The recognition of new limitations that will presumably be assigned to future working quantum computers is one such problem. This is related to the notions of randomness and classes of uncomputability that would characterize the new limits for quantum computations. True random sequences of numbers are considered limits for classical computers, as only pseudo-random numbers lie within reach for them, whereas these true random sequences could be realized by future quantum computers, e.g. Orts et al.Orts et al. 2023. Thus, a natural question is whether there are new limitations to this quantum randomness. More precisely, what is the hierarchy for quantum randomness envisaged by different QRNGs, and what is the common limiting randomness which would transcend all quantum random sequences but still remain within the scope of QM? Even though such questions seem to be of purely theoretical interest at present, they may serve to bootstrap the entire effort of seeking to understand ways in which QM could be extended. This could prove important when searching for an appropriate formalism such as would be required by a reconciliation of QM with GR. We are not deciding here that such limits are now in theory well defined, or capable of being accessed in the laboratory. Nevertheless, Tsirelson's conjecture, especially in terms of how it is formally negated, furnishes a motivation for initiating theoretical work on the topic of their accessibility. This is somewhat similar to considering extensions of QM in terms of Bell-style bounds, where no-signaling extensions are considered with bounds exceeded by $2\sqrt{2}$, e.g. ref. Masanes, Acin, and Gisin 2006. As the analysis in this paper shows, the infinite families of quantum measurements can amount to a delicate formal tool that enables one to distinguish TC from $\neg$TC on the one hand, and an understanding of randomness in QM on the other. However, this requires much care and insight when defining and manipulating such infinite constructs from different branches of mathematics. Thus, an understanding of Tsirelson's problem points to the formal side of QM that might be useful in respect of the task of its proper extension over infinite constructs. One deep result is that the Born probabilistic rule for a single measurement defines the probabilistic measure over the spaces of infinite sequences of outcomes of QM experiments, and determines the 1-randomness of the sequences (e.g. Landsman 2020 and Subsection 4.4). We believe that the problems with the extension of QM over infinities of modes or particles, as in quantum field theory, also require careful reconsideration in line with the methods adopted in this paper. This connection with quantum field theory and general relativity will be the topic of a separate forthcoming work by the present authors.

Let $\sigma_0 \in 2^{<\omega}$ be a binary finite sequence of outcomes of the repeating quantum mechanical yes-no measurements on the 2-dimensional Hilbert space $\mathcal{H}^{(2)} \simeq \mathbb{C}^2$, where $\mathbb{C}$ is the field of the complex numbers. Such a sequence is realized, e.g., when measuring the $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Pauli matrix in the normalized state $1/\sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, but the quite similar 'classical' process of tossing a symmetric coin cannot in fact be realized deterministically and so classically. Such finite repetitions can be extended towards infinite binary sequences $\sigma \in 2^{\omega}$ of the outcomes of repeating quantum measurements (e.g., ref. Landsman 2020). One could wonder whether there is any sense to such an extension up to infinity of the number of potential experiments considered within QM. On the one hand, this can be seen as a theoretical construct allowing one to test the borders of the applicability of the finite dimensional Hilbert spaces and the finite numbers of eigenfunctions and eigenvalues which are typically accessible within type I von Neumann algebras of observables. On the other hand, the infinite-dimensional cases are the part of conventional QM on Hilbert spaces, and the Born rule for a single run of quantum measurement already contains information about the probability of the whole run of infinite repetitions of the measurements. Thus, technically, the infinite case is already present in the standard formalism of QM.

The measurement is now performed on the entire Hilbert space $\mathcal{H} = (\mathcal{H}^{(2)})^K = \bigotimes_{i=0}^{K} \mathcal{H}_i^{(2)}$ in the $K < \infty$ repetitions case, or $\mathcal{H} = (\mathcal{H}^{(2)})^{\omega} = \otimes_{i\in\omega}\mathcal{H}_i^{(2)}$ for the infinite sequences of outcomes. The important observation made in ref. Landsman 2020 is that there is the equivalence (I_{AB}) between the following two procedures

A. s is generated in measurement performed as the 'whole run system' with $H^{\infty} = \bigotimes_{i=1}^{\infty} \mathcal{H}_i^{(2)}$, and then the statistical results on the ensemble of ss determine the probability measure P^{∞} on $\mathcal{H}^{\infty}$. One considers the 'whole run system' a quantum system on which there are performed measurements.
B. s is retrieved by collecting the statistical results at each ith, i.e. performed on $\mathcal{H}_i^{(2)}$, and thus drawing conclusions about the limiting statistical probability of the sequences in 2^{ω}.

As different as these situations may seem, they are nevertheless equivalent. This, again, is basically due to Landsman 2020, Theorem 5.1. The point is the Born rule, namely the fact that P^{∞}, is precisely P^{∞}_{Born} which, in turn, is uniquely generated by the single Born probability $P_{i,Born}$ Landsman 2020; Król, Bielas, and Asselmeyer-Maluga 2023. This point is also addressed in the supplementary subsection 4.4, where quantum states are explicitly referred to. We want to explore this equivalence further from the computational complexity point of view. Our aim is to obtain certain classes of infinite binary sequences which might distinguish two Hilbert spaces in QM, and the correlations of observables on them, which appear in the formulation of Tsirelson's conjecture (TC). So at first we claim that *there exist infinite binary sequences on infinite dimensional Hilbert spaces capable of distinguishing the correlations as in Tsirelson's conjecture on* $\mathcal{H}^{(\infty)}$ *and* $\mathcal{H}_A \otimes \mathcal{H}_B$.

Let us turn again to the equivalence I_{AB} and the infinite tensor product (ITP) of finite-dimensional Hilbert spaces $\mathcal{H}^{(\infty)}$. There is one general point connected with the ITP of finite-dimensional Hilbert spaces, which even for the minimal complex case $\dim(\mathcal{H}_i) = 2, i = 0, 1, 2, \ldots$, leads to a non-separable Hilbert space. This has already been nicely analysed by John von NeumannVon Neumann 1939 (see also Thiemann and Winkler 2001), who showed that the Hilbert space $\mathcal{H}^{(\infty)}$ assigned to the countably infinite tensor product $\mathcal{H}^{(\infty)} = \bigotimes_{i=1}^{\infty} \mathcal{H}_i^{(2)}$ must be a non-separable infinite-dimensional one: the (complete) infinite tensor product of countably infinitely many, at least 2-dimensional, Hilbert spaces is *nonseparable* infinite-dimensional $\mathcal{H}^{(\infty)}$. However, as is shown by Landsmann, this does not affect the equivalence I_{AB}, which remains true Landsman 2020. One might think that the ordering in the sequences of Hilbert spaces matters, but it has also been shown by von Neumann that the resulting $\mathcal{H}^{(\infty)}$ does not depend on any ordering of infinitely countable sets of indices over which the tensoring is taken.

Thus we adopt a different perspective on the sets of indices, such that various Turing uncomputability classes might distinguish them. At the basic level, the set I of indices is to be *recursively enumerable* (r.e.), thus leading to r.e. sequences of spaces and the Hilbert space $\mathcal{H}^{(\infty)}$. The uncomputable power of sequences of indices is then enhanced upwards to the higher uncomputability Turing classes, thus enriching the structure of the infinite tensor products of Hilbert spaces. Computationally, there are different ITPs, $\mathrm{ITP}_{\mathbf{a}}$ being where $\mathbf{a}$ is the Turing class of the sequence of indices $\sigma = (i_1, i_2, \ldots, i_k, \ldots), k = 1, 2, \ldots$. The resulting infinite-dimensional Hilbert spaces $\mathcal{H}_{\mathbf{a}}^{(\infty)}$ are again non-separable, as whenever there exists a countable base for $\mathcal{H}$ there will exist a r.e. procedure for generating its subspaces spanned on the subsets of the vectors from a base, and the entire $\mathcal{H}$. Again, any direct application of the Turing classes to the sets of indices will not affect the procedure of ITP, nor the equivalence I_{AB}. One can also consider other definitions of ITPs, such as the inductive limit of finite-dimensional Hilbert spaces (A. Guichardet, 1969) or the approach involving grounded products of Hilbert spaces (J. Baez, 1993). These ITPs leads directly to separable Hilbert spaces, and from that point of view are suitable for certain physical applications; see also Subsection 4.4.

Next we will be allowing for random binary sequences of indices, and studying their influence on the ITP construction. This should be considered a purely theoretical tool, without imposing at this stage any observable effects. Even though one could not expect a big departure from the cases discussed so far, there are certain limitations to the applicability of random sequences together with all sequences from higher Turing classes, and the limitations are analysed in algorithmic randomness theory (e.g. ref.Downey and Hirschfeldt 2010). The obstructions appear in a pure form in higher random and higher Turing degrees. Instead of the word 'generic' that has usually figured in the algorithmic randomness literature (e.g. ref. Downey and Hirschfeldt 2010), we will be using the expression '(C)-generic' (Cohen generic) here, while 'randomness' will be understood as Martin-Löf (ML) algorithmic randomness (n-randomness, $n = 1, 2, \cdots$), e.g. ref. Downey and Hirschfeldt 2010. There is a set of important results connecting both kinds of infinite binary sequence: i.e. (C)-generic and random. The point is that the sequences are orthogonal in a special sense, in the entirety of Turing classes above 2-random and 2-generic. This is precisely the content of the following results:

Theorem 1 (Nies, Stephan, and Terwijn 2005) *Any 2-random sequence and any 2-(C)-generic sequence always form a minimal pair in the Turing degrees.*

We say that a Turing degree $\mathbf{b}$ bounds $\mathbf{a}$ when $\mathbf{a} \leq \mathbf{b}$ and this is when $\forall_{\sigma_{\mathbf{a}} \in \mathbf{a}} \forall_{\sigma_{\mathbf{b}} \in \mathbf{b}} \sigma_{\mathbf{a}} \leq_T \sigma_{\mathbf{b}}$, where $\leq_T$ is the Turing order relation between sets $\sigma_{\mathbf{a}}, \sigma_{\mathbf{b}}$. *A minimal pair* $(\mathbf{a}, \mathbf{b})$ of Turing degrees $\mathbf{a}$ and $\mathbf{b}$ for the random and generic sequences, is when the only degree $\mathbf{c} < \mathbf{a}$ and $\mathbf{c} < \mathbf{b}$ in the c.e. class is $\mathbf{c} = \mathbf{0}$. Then it holds

Theorem 2 (Kurtz 1981, Kautz 1991) *Every 2-random degree bounds a 1-(C)-generic degree.*

So, in the lower degrees, where $n \leq 2$, the randomness and (C)-genericity mix correspondingly, as in Theorem 2 above. For higher degrees, however, over and above 2-degree, an n-generic sequence will never be n-C-random and an n-random sequence will never be n-(C)-generic where $n > 2$, as follows from Theorem 1. This means that there is no nontrivial 'intersection' of $\mathbf{a}$ and $\mathbf{b}$ in the partial order of degrees. The above relation of random and (C)-generic sequences and their orthogonality in higher degrees offers a clue in respect of our search for tools capable of distinguishing TC and $\neg$TC.

There are, in general, two notions of genericity characterizing arithmetical and random sequences (e.g. refs. Downey and Hirschfeldt 2010; Kautz 1991). One is Cohen genericity, which is related to Cohen forcing in set theory, and here it is applied to Peano arithmetic (PA), while arithmetic randomness is closely related to Solovay genericity, which is also the specialization of the *random forcing* known from set theory to PA. Such specialization of set theory forcings to arithmetic is called *miniaturisation* Downey and Hirschfeldt 2010; Kautz 1991 and is based on ideas in ref. Fefferman 1964. Here we are performing a kind of *deminiaturisation*, which relies on extending the arithmetical perspective to set theory (ZFC) and relating this to QM on infinite dimensional Hilbert spaces (e.g. ref. Król, Bielas, and Asselmeyer-Maluga 2023). In particular, the arithmetic orthogonality of C-generic and random sequences is elevated to Cohen and random forcings in a certain model M of ZFC. It is known that given the forcing random extension $M[r]$ (where r represents a random generic real number), it does not contain any generic Cohen sequences (reals) amongst the random generics, and the converse also holds true. Based on the formal structure of the lattice of projections $\mathbb{L}(\mathcal{H}^{(\infty)})$ on the infinite-dimensional separable Hilbert space, one can infer that the structure will favor random forcing under certain conditions. Thus applying this bipartite exclusion principle of randomness and genericity for M and making use of some Boolean-valued model theory, one can find distinctions between the correlations on the product $\mathcal{H}_A \otimes \mathcal{H}_B$ from those on $\mathcal{H}$. It follows that there exist some correlations on $\mathcal{H}^{\infty}$ predicted by QM which can not be reproduced from those on $\mathcal{H}_A \otimes \mathcal{H}_B$.

Let us now briefly discuss the main findings of this work. 1. Given the equivalence I_{AB} from the beginning of this section, we show that it is actually stratified for QM on infinite dimensional Hilbert spaces – something which will depend on our way of constructing the infinite tensor product, specifically allowing for ZFC-generalized sequences of Hilbert spaces and observables. When one performs the measurements on the whole-run-system of such ZFC-generalized products, and compares this with the outcomes obtained 'locally' on the ith entry of the product, the results in the two cases can be statistically different. We find that in ZFC-generalized ITP (random generic) products there has to appear, 'generically,' a self-adjoint operator A that locally is absent from the collecting data. The operator A envisages a distinction between collecting local statistical outcomes and measurements on the whole-run-system on $\mathcal{H}^{\infty}$. Based on this, we can distinguish TC from $\neg$TC. On the other hand, local statistical information derived from the sequence (at ith entry) cannot distinguish TC and $\neg$TC, but rather reproduces the TC paradigm. This is not any direct negation of the equivalence I_{AB}; rather we are indicating generalized infinite tensor products (which are not the von Neumann ones) that differ with respect to such statistical predictions. 2. We carefully distinguish between the sequences of outcomes on $\mathcal{H}_a \otimes \mathcal{H}_b$ and on $\mathcal{H}^{\infty}$, showing directly, in particular, that the latter can be more correlated than the former. The finite refinement of this leads to the negation of TC. 3. We thus extract (following ref. Król, Bielas, and Asselmeyer-Maluga 2023) the appropriate form of randomness for infinite sequences on $\mathcal{H}^{\infty}$, which is ZFC-genericity (or Solovay set theory genericity) as distinct from pure arithmetic genericity (miniaturization to PA). QM (on infinite dimensional Hilbert spaces) leads to ZFC genericity, which requires the formal perspective of a model of ZFC. We consider this the fundamental feature of QM. 4. We discuss the findings from the perspective of their eventual applicability or practical use. We also discuss the problem of the eventual detection of ZFC-generic random sequences of QM outcomes and distinguishing these from arithmetic generic ones, where in practice this could lead to distinguishing TC from $\neg$TC.

The paper is organized in such a way that the central result is Theorem 4 in the Results section; however, its proof contains several stages encapsulated in lemmas, proposition and remarks that involve a presentation and discussion of set theory genericity in QM along with the above-mentioned issues. We close the main body of the paper with a Discussion and Supplementary Material section including various aspects of the extended QM formalism over set-theory component, measurements in such extended QM, ITPs, and Subsection 4.1 contains the proof of Proposition 1, which is a vital part of the proof of Theorem 4. The next section is the Key Terminologies section. The approach developed here mimics to some degree the Gödel incompleteness theorem: for sufficiently rich theories having ω-enumarable sets of axioms, we will always encounter some true sentences that remain unprovable in such axiomatized theories. The point is that when one takes higher than r.e. classes of axioms, the proofs can in general be attained and the 1st incompleteness theorem of Gödel does not hold. On our approach, we have the Turing enumerable tensor products of finite dimensional Hilbert spaces, which cannot approach sufficiently closely certain infinite-dimensional Hilbert spaces that can still figure in QM considerations. Then the whole-run-system measurements on Hilbert spaces like this lead to sequences allowing for a differentiation between product and nonproduct cases. The formal methods used here are therefore based on models of ZFC and, in particular, on the Boolean-valued models that, since the 1970s, have been developed in the context of QM by Gaisi Takeuti and Masanao Ozawa (e.g. refs. Takeuti 1978; M. Ozawa 2021).

2. Key terminologies

2.1 Turing uncomputable ITPs of Hilbert spaces

We refer Readers to ref. Soare 2016 for all additional information regarding Turing computability and uncomputable classes and to ref. Downey and Hirschfeldt 2010 regarding their relation to randomness. The supplementary subsection 4.4 also contains discussion of ITPs. Let $2^{<\omega}$ be a partial order of finite binary sequences and A a set. $A^{<\omega}$ is a set of all finite subsets of elements of A. Let A be some set of finite dimensional Hilbert spaces. Given $\sigma \in A^{<\omega}$ let $\bigotimes \sigma$ be $\bigotimes_i \sigma(i) = \bigotimes_{i\in[n]} \mathcal{H}_i$, $i = 1, 2, \dots, n$; $\sigma = (\sigma_{j_i}, i = 1, 2, \dots, n)$, $j_i \in \mathbb{N}$ and $\sigma_{j_i} = \sigma(i) = \mathcal{H}_i$ and $\dim(\mathcal{H}_i) = j_i$. Here $[n]$ is the finite sequence $[n] = (1, 2, \dots, n)$. It certainly holds $\bigotimes \sigma = \mathcal{H}_\sigma$ is a finite dimensional Hilbert space and $\dim(\mathcal{H}_\sigma) = j_1 \cdot j_2 \cdot \ldots \cdot j_n$. We are interested in the process of attaining $\mathcal{H}^{\infty}$ an infinite dimensional Hilbert space, from finite dimensional Hilbert spaces in A. In general there are ITPs which govern this kind of successive up to infinity tensor products. This is considered as 'computationally trivial' in a sense that the sets of indices are Turing computable or computably enumerable (c.e.) e.g. ref. Soare 2016. Allowing for the uncomputable classes does not change much with the resulting ITPs (see Proposition 1 in the Results section). However, existence of such classes is the reason that not all sequences of Hilbert spaces are equally well-accessible from c.e. classes. Thus forming sequences of Hilbert spaces which are uncomputable, i.e. the sequences of their indices are represented by binary uncomputable sequences, and then tensoring collectively the Hilbert spaces with indices from this uncomputable sequence, is out of reach from trivial c.e. sequences of such spaces by *computable* processes. In principle, one could reach computationally such sequences by oracle Turing machines and oracles have to be 1-random (e.g. Martin Löf 1-random, ref. Downey and Hirschfeldt 2010) to guarantee an universal access to arbitrary sequences (such an 1-random oracle is uncomputable by itself)

Theorem 3 (Kučera 1984, Gács 1986) *Any sequence is computed by some 1-random sequence.*

However, by no means such 1-random sequence is unique for all sequences: there does not exist any single 1-random sequence computing all others. In contrary, there are many highly incomparable fractions of uncomputable sequences. In the Results section we will make use and elaborate over fraction of random vs. (C)-generic sequences and their mutual incompatibility (in higher degrees) in the context of the Tsirelson's conjecture.

Let us be more specific and as the example consider the spaces $2^{\omega} \times \mathcal{H}^{(2)}, 2^{<\omega} \times \mathcal{H}^{(2)}$ so thus they comprise sequences of 2-dimensional complex Hilbert spaces. If $\sigma \in 2^{\omega} \times \mathcal{H}^{(2)}, (2^{<\omega} \times \mathcal{H}^{(2)})$ then $\bigotimes \sigma$ is the infinite (finite) tensor product of the spaces. Even though the separable Hilbert spaces of the same dimension are isomorphic they can differ by the complexity of the sequences of the indices numbering the spaces. They are binary sequences belonging to 2^{ω} or $2^{<\omega}$. Consequently the spaces comprise the sequences belonging to different Turing classes, even though they number collection of identical Hilbert spaces like $\mathbb{C}^2$. Let $\mathbf{a}_s$ be the Turing class of a binary sequence s.

i. The tensor product $\bigotimes \sigma$ of a sequence $\sigma \in 2^{\omega} \times \mathcal{H}^{(2)}$ inherits the complexity of the binary sequence of its indices, $s \in 2^{\omega}$, and thus σ is in the same Turing class as s, i.e. $\mathbf{a}_{\sigma} \coloneqq \mathbf{a}_s$.
ii. An infinite dimensional Hilbert space $\mathcal{H}^{\infty}$ is attained in the Turing class $\mathbf{a}$ if there exist $\sigma \in A^{\omega}$ such that $\bigotimes \sigma = \mathcal{H}^{\infty}$ and $\mathbf{a}_{\sigma} = \mathbf{a}$.
iii. The pair $(\mathcal{H}^{\infty}, \sigma)$ is called $\mathbf{a}_{\sigma}$ $\mathcal{H}^{\infty}$ or $\mathcal{H}^{\infty}_{\mathbf{a}}$ Hilbert space. So $(\mathcal{H}^{\infty}, \sigma) = (\mathcal{H}'^{\infty}, \sigma')$ if not only $\mathcal{H}^{\infty} \overset{iso}{=} \mathcal{H}'^{\infty}$ as Hilbert spaces but also $\mathbf{a}_{\sigma} = \mathbf{a}_{\sigma'}$ in a sense that for any attainability class $\mathbf{a}_{\sigma}$ of $\mathcal{H}^{\infty}$ one finds equal to it the attainability class $\mathbf{a}_{\sigma'}$ of $\mathcal{H}'^{\infty}$ and conversely.

Remark 1 *In general, considering $\mathcal{H}$ as isometric to $\mathcal{H}'$ does not require fixing the computational classes of the spaces. However, the pairs $(\mathcal{H}^{\infty}, \sigma)$ clearly respect the Turing classes $\mathbf{a}_{\sigma}$ within which $\mathcal{H}^{\infty}$ has been generated as $\otimes\sigma$. So, formally, Turing uncomputability classes can matter as additional parameter.*

It might seem that appearence of uncomputable sequences of Hilbert spaces does not contribute much into considerations where the computability questions are not in the main scope of the analysis. However, we will see that certain refinements of the uncomputable sequences (ZFC twist) already determine the limits to the Landsman's equivalence and shed light on the Tsirelson's Conjecture. In what follows in general we are using $\mathcal{H}^{\infty}$ for a separable infinite dimensional complex Hilbert space and $\mathcal{H}^{(\infty)}$ for nonseparable infinite dimensional complex ones (like in the von Neumann's complete ITP). We are using the symbol σ as representing also binary sequences from 2^{ω} (not only the sequences of Hilbert spaces as above) but the use is clear from the context and should not lead to any misunderstanding.

2.2 The Tsirelson's Conjecture

The TC as QM problem can be expressed as follows ref. Scholz and Werner 2008 – more detail presentation is given in the Results section in the proof of Theorem 4. Let a, b be two independent observers conducting measurements over a quantum system S in $\mathcal{H}$ which is the Hilbert space of states of S. There are two possible scenarios realising the idea of independence of such independent measurements of a and b on $\mathcal{H}$. One scenario assigns $\mathcal{H}_a, \mathcal{H}_b$ two Hilbert spaces of states to a, b such that the joint system is described by $\mathcal{H}_a \otimes \mathcal{H}_b$ and clearly the measurements can be performed independently on each factor. However, noticing that independence enforces the commutativity of the observables of a with those of b, instead of factorizing of $\mathcal{H}$ into the tensor product one can perform measurements on the entire $\mathcal{H}$ under the supposition that the observables $\mathcal{A}$ and $\mathcal{B}$ commute as operators on $\mathcal{H}$. This is roughly the second scenario. The Tsirelson's problem (still roughly) is the statement deciding whether these two situations always lead to the same sets of correlation functions. TC says they are always equivalent, in a sense that given arbitrary $\mathcal{H}$ as in QM there always exist two $\mathcal{H}_a, \mathcal{H}_b$ and the 1st scenario on $\mathcal{H}$ is always equivalent to some 2nd on $\mathcal{H}_a \otimes \mathcal{H}_b$. The theorem already proved by Tsirelson is that for finite dimensional Hilbert spaces the two situations indeed give rise to the same sets of correlation functions leaving open, up to recently, the infinite dimensional case. The affirmative solution of $\neg$TC on infinite dimensional Hilbert spaces has been given in ref. Zhengfeng Ji et al. 2022 by employing in particular model theory and Turing uncomputability classes the tools we are also using in this work. We think the tools are not accidental but essential for the QM formalism leading to a new understanding of QM.

Given an infinite dimensional separable complex Hilbert space $\mathcal{H}$ there always exist an isometric isomorphism between it and the Hilbert space ℓ_2 of square-summable infinite sequences of complex numbers. Let $V : \ell_2 \to \mathcal{H}$ be such an isometry and $V^* : \mathcal{H} \to \ell_2$ its adjoint. Following ref. Zhengfeng Ji et al. 2022 let the simplified measurement be given by projective valued measures (PVM) i.e. by the collection of projections $\{P_i^k\}, i = 1, 2 \cdots, m$, $\sum_{i=1}^m P_i^k = \mathbb{1}$ and there are n such POVs, i.e. $k = 1, 2 \cdots, n$ for a. Similarly there are n many PVMs for b observer i.e. $\{Q_j^l\}, j = 1, 2 \cdots, m$, $\sum_{j=1}^m Q_j^k = \mathbb{1}$ and $l = 1, 2 \cdots, n$. There results the set of correlations (see (5) and (6) in the Results section for the explicit use of states) between these POVs of a and b on $\mathcal{H}$ (assuming the commutativity $[P_i^k, Q_j^l] = 0, i, j = 1, 2, \cdots, m; k, l = 1, 2, \cdots, n$):

$$\{Corr(\mathcal{A} \cdot \mathcal{B})_{ij}^{kl}\} = \{V^* P_i^k \cdot Q_j^k\ V\}.$$

On the product space $\mathcal{H}_a \otimes \mathcal{H}_b$ an isometry V reads now $V : \ell_2 \to \mathcal{H}_a \otimes \mathcal{H}_b$ with $V^* : \mathcal{H}_a \otimes \mathcal{H}_b \to \ell_2$ and the PVMs of a are now $P_i^k \otimes \mathbb{1}_b$ and PVMs of b $\mathbb{1}_a \otimes Q_j^k$ $i, j = 1, 2, \cdots, m; k, l = 1, 2, \cdots, n$. The corresponding set of correlations now reads

$$\{Corr(\mathcal{A} \otimes \mathcal{B})_{ij}^{kl}\} = \{V^* P_i^k \otimes Q_j^k\ V\}.$$

Then the Tsirelson's conjecture is the statement that for any such settings and any $m, n \in \mathbb{N} \setminus \{0\}$ the sets of correlations $Comp(\{Corr(\mathcal{A} \otimes \mathcal{B})_{ij}^{kl}\})$ and $\{Corr(\mathcal{A} \cdot \mathcal{B})_{ij}^{kl}\}$ are the same, where $Comp(A)$ is the topological completion of the set A.

The measurements in QM of a quantum system S are formally represented by more general positive operator valued measures (POVM) which extends projection valued measures assigned to pure states, such that POVM can comprise also mixed states ρ.

3. Results

Let us take a closer look at the Landsman's equivalence from the Introduction from the point of view of generalised products from different Turing classes. One can stratify directly the equivalence in a way which does not affect it: given the equivalence of A and B, the reduction of A to B is not necessary any r.e. process, so that the higher Turing degrees of the products of Hilbert spaces are allowed.

A'. s is generated in a measurement performed on the 'whole run system' with $\mathcal{H}^{(\infty)} = \bigotimes_{s_i}^{\infty} \mathcal{H}_{s_i}^{(2)}$, however, $s = \{s_i, i \in \mathbb{N}\}$ is not r.e. which means it is in a higher Turing class. Then the statistical results on the ensemble of ss determine the probability measure P^{∞} on $\mathcal{H}^{(\infty)}$. One considers the 'whole run system' as a quantum system on $\mathcal{H}^{(\infty)}$ on which there are performed measurements.

A. s is generated in a measurement performed on the 'whole run system' with $\mathcal{H}^{(\infty)} = \bigotimes_{i=1}^{\infty} \mathcal{H}_i^{(2)}$ with r.e. sequence of Hilbert spaces η and then, the statistical results on the ensemble of ss determine the probability measure P^{∞} on $\mathcal{H}^{(\infty)}$.

B. s is retrieved by collecting the statistical results at each ith, i.e. performed on $\mathcal{H}_i^{(2)}$, and thus concluding about the limiting statistical probability of the sequences in 2^{ω}.

A' is not Turing reducible to B since the preparing of states procedures in B are understood as given by r.e. procedures. Thus more precisely, when A' contains non r.e. sequences it might be nonequivalent to B. but also nonequivalent to A., if one assumes the equivalence of A. and B.. So far we have a direct relativisation to different Turing classes which follows the obvious distinctions

$$(\mathcal{H}^{(\infty)}, \eta) \not\simeq (\mathcal{H}^{(\infty)}, \eta') \text{ i.e. } \mathbf{a}_{\sigma(\eta)} \not\simeq \mathbf{a}_{\sigma'(\eta')} \text{ and } \mathcal{H}^{(\infty)} = \bigotimes \eta, \mathcal{H}^{(\infty)'} = \bigotimes \eta'.$$

Here the sequence $\sigma(\eta)$ is the binary infinite $(0, 1)$ sequence of indices of the sequence η of finite dimensional Hilbert spaces.

Still the consequence regarding the 'absolute' irreducibility of A' and B is not decided and needs a justification. As we have noted already in the Introduction the procedure of hyper-Turing tensoring the sequences of Hilbert spaces leads as a rule to presumably a nonseparable infinite dimensional space as does any ITP of the Hilbert spaces of dimension at least 2. Still it holds

Proposition 1 *Allowing for generalised Turing uncomputable products of Hilbert spaces the collecting statistical data on the whole-run-system as in A' is equivalent to the step-by-step collecting statistical data as in B which is equivalent to A.*

This proposition follows from the Landsmann's result supporting the statistical equivalence of ITP in A with the single turns performed on the ith Hilbert space in the product and from the von Neumann's result showing the invariance of ITP to different arbitrary orderings of the countable infinite sets of indices of the spaces in ITP $\mathcal{H}^{(\infty)}$Von Neumann 1939.

Remaining in the countable infinite number of factors in ITP of Hilbert spaces can we testify the limits of the equivalence A and B in QM? This is what we want to explore now by defining the *ZFC-twisted ITP* of Hilbert spaces.

Remark 2 *Turning to ZFC is based on two important observations. One follows from Theorem 3 from the Key terminologies section stating that the most general perspective on uncomputable sequences requires random sequences which are related with random forcing. The other observation is that the generalisation of the arithmetic forcings reduced to PA is the extension to ZFC random and Cohen forcings. Thus we fix a general perspective as this assigned to ZFC.*

So we refer to ZFC random sets of indices in ITPs. The question arises: are we still within the range of QM? In ref. Król, Bielas, and Asselmeyer-Maluga 2023 it was shown that QM on infinite dimensional Hilbert spaces is indeed set theory Solovay generic, i.e. ZFC-random. This means that certain binary sequences of QM outcomes were formally described as 'generic random real numbers' known from set theory. This requires, however, certain additional suppositions. So thus following ref. Król, Bielas, and Asselmeyer-Maluga 2023 let us assume the working criterion for ZFC randomness of QM:

> [ZFC random] QM on $\mathcal{H}^{(\infty)}$ is ZFC random when there exists a binary random Solovay generic sequence of outcomes, represented by $r \in \mathbb{R}$, generated by the maximal complete atomless Boolean algebra of projections B in the lattice of projections $\mathbb{L}(\mathcal{H}^{(\infty)})$. To be generated by the algebra B means that r is generic over B in V or, perhaps, in some other transitive, standard model M of ZFC.

We refer the reader to Subsection 4.5 for a more detail analysis of the ZFC twist of QM from the point of view of the standard postulates of QM.

Remark 3 *B becomes internal algebra in V (M) and thus the randomness of $\sigma \in 2^{\omega}$ is also reduced to V or M, where 2^{ω} is now $2^{\omega}_{|V}$ or $2^{\omega}_{|M}$.*

We want to understand the randomness of $\mathbb{R} \ni r = \sigma \in 2^{\omega}$ for the Tsilerson's conjecture by allowing for two stages of randomness for infinite sequences of QM outcomes: one is the randomness in the product of $\mathbb{C}^2$ Hilbert spaces, where at each ith entry there is a r_i outcome resulting from the measurement procedure performed on $\mathbb{C}_i$. The other stage of randomness is given by the set of indices $\{i_k\}_{k\in\mathbb{N}}$ which is the binary sequence $\sigma \in 2^{\omega}$ which can be Solovay random (relative to a model M).

Remark 4 *Given an infinite set of real numbers $\{r_{i_k}, k \in \mathbb{N}, \}$, with Solovay random indices $\{i_k\}_{k\in\mathbb{N}}$ the set $\{r_{i_k}\}$ is a random sequence of real numbers.*

It follows that we do not need to refer to any quantum process for generating the real entries of the sequence $\{r_{i_k}, k \in \mathbb{N}, \}$ and can focus on purely random or uncomputable properties of its index set. Thus let us introduce the ZFC randomness into the strata of ITPs. The procedure relies on the ITPs $\mathcal{H}^{(\infty)} = \bigotimes \eta(\mathbb{C}^2)$, however, the binary sequence $\sigma(\eta) \in 2^{\omega}$ of indices is now considered *ZFC random*. This generalises (or twists) the arithmetic ML n-random sequences. ML n-randomness requires just passing the nth ML test, e.g. ref. Downey and Hirschfeldt 2010, which is fulfilling certain arithmetical properties by $\sigma(\eta) \in 2^{\omega}$. This does not refer to any model of ZFC, similarly the C-genericity and arithmetic Solovay genericity do not refer to such models. The next step, however, requires performing the *ZFC twist* as in Remark 2 and choosing a ZFC model extending arithmetic approach appropriately. This is a kind of broader formal perspective (models of ZFC) which is to be included into the analysis, since the perspective of axiomatised formal PA does not suffice in particular for addressing QM as ZFC random.

Proposition 2 *Let $\sigma = \{i_k\}_{k\in\mathbb{N}} \in 2^{\omega}$ be a ZFC random binary sequence of indices relative to a transitive standard ZFC model M. Then $\bigotimes_{i_k}^{\infty} \mathbb{C}^2_{i_k}$ determines the ITP $\mathcal{H}^{(\infty)}$ and certain s.a. operator $\mathcal{A}$ on a separable infinite dimensional Hilbert space $\mathcal{H}^{\infty}$.*

For the proof see supplementary subsection 4.1.

Corollary 1 *Let $\{a_{i_k}\}, k \in \mathbb{N}$ be a sequence of s.a. commuting operators on $\mathbb{C}^n$ and $\{i_k\}_{k\in\mathbb{N}}$ be a ZFC random binary sequence of indices. Then taking the tensor product $a_{i_1} \otimes a_{i_2} \otimes \cdots$ gives rise to $a \oplus \mathcal{A}$ where a is a s.a. operator on $\mathcal{H}^{(\infty)}$ and $\mathcal{A}$ is s.a. operator on $\mathcal{H}^{\infty}$.*

We will see in what follows that the appearance of operators like $\mathcal{A}$ above may help distinguishing TC from $\neg$TC on $\mathcal{H}^{\infty}$. We are proving the following result

Theorem 4 *If QM is ZFC random then the Tsirelson's conjecture fails.*

To show this let us start with general remarks. ZFC randomness of QM does not mean that all random sequences generated by quantum measurements lead to ZFC random sequences. Instead this is rather an option realised in some conditions. It follows that there might be infinite ZFC random sequences of outcomes as well ML n-random ones or, in general, some other certified differently. But we can always refer to the option and say 'let σ be a ZFC random sequence of QM outcomes on $\mathcal{H}^\infty$' and then analyse the conclusions of the existence of such σ.

Thus as defined in [ZFC rand] QM supports the existence of a Solovay generic random binary sequence $r \in \mathbb{R}$ of QM outcomes. This requirement has already strong formal implications. On the one hand the existence of random genericity in QM refers to models of ZFC, which has been partly recognised in ref. Król, Bielas, and Asselmeyer-Maluga 2023. On the other hand this is a general question regarding the existence of generic reals in models of set theory and this fixes the perspective into more formal. One proves in ZFC that the generic filters G for the algebra B in M exist for a CTM M so thus Solovay generic reals added by random forcing to M (note that B - the maximal Boolean algebra of projections in $\mathbb{L}(H^\infty)$ is an *atomless* complete measure algebra). For Boolean-valued models, M^B, of ZFC such generic reals r exist with Boolean value 1 (provided B being atomless in M). Moreover, the 2-valued model of ZFC (the forcing extension of M) containing the reals r, $M[G]$, are obtained as the quotient model, i.e. $M^B/G = M[G]$. For the case of universe V of set theory one defines the Boolean valued model V^B in V (but still the canonical embedding $V \hookrightarrow V^B$ exists) and generic reals do exist with the Boolean value 1 (see the supplementary subsection 4.1). Such approach is also the resolving of the issue of nonexistence of generic filters in V such that, from the point of view of the cumulative universe of sets V considering generic random sequences requires referring to a CTM M or to Boolean-valued models like V^B.

Remark 5 *There are various results in set theory which enable genericity, e.g. allowing for the negation of the continuum hypothesis (CH) it is possible (let $\mathfrak{c}$ represents continuum) that there are cardinalities κ, $\aleph_0 < \kappa < \mathfrak{c}$, for which generic filters exist and there are Martin's axioms $MA(\kappa)$ stating this ($MA(\kappa)$ are independent of ZFC and consistent with $\neg CH$). There exist also the refined versions of forcings fulfilling variety of forcing axioms, like proper forcing, enabling the genericity. At this stage we do not think that the refined results as above should be addressed in the context of QM currently.*

The proof goes from this standpoint of the universe V.

Thus let us turn to a CTM M which supports the existence of r generic over B in M, which guarantees that QM be ZFC random. We are building the binary sequences of indices for which there correspond s.a. operators on $\mathcal{H}$ which additionally may (or may not) be correlated with the operator $\mathcal{A} = \mathcal{A}_r$ from the Proposition 2.

The fundamental property of generic extensions related to a CTM M says that (e.g. ref.Bartoszyński and Judah 1995)

Lemma 1 *For any random generic $r \in M[r]$ there does not exist any Cohen generic $c \in M[r]$ such that $r = c$. For any Cohen generic $c \in M[c]$ there does not exist any Solovay generic $r \in M[c]$ such that $c = r$.*

Remark 6 *'Generic' here means random or Cohen reals not in the ground model M but in $M[r]$ or $M[c]$ correspondingly and such generic reals correspond to the generic filters in the complete random, B, and Cohen, C, Boolean algebras in M. Reals in a model M are those binary sequences which are in 2^ω in M, i.e. 2^ω_M. Thus the lemma states that among random generic reals in the extension there are no Cohen generic, and conversely.*

We have two results

Lemma 2 *In V^B real numbers are in 1:1 correspondence with s.a. commuting operators in B on $\mathcal{H}^\infty$.*

Lemma 3 *Let c be a Cohen and r a random generic reals, $R_{M[c]}$ the reals in $M[c]$ and $R_{M[r]}$ the reals in $M[r]$. Then it holds: $c \in R_{M[c]} \subset \mathbb{R}$ and $r \in R_{M[r]} \subset \mathbb{R}$ and $\mathbb{R} \hookrightarrow R_{V^B}$.*

Lemma 2 is the direct conclusion from Lemmas 1 and 2 in the SM A file. Lemma 3 above recapitulates the elementary relations for CTMs and Boolean models of ZFC Jech 2003; Bell 2005.

So let M be the standard transitive ZFC model (possible countable) according to which $r = \{i_k\}_{k\in\mathbb{N}}$ is generic random and $c = \{j_k\}_{j\in\mathbb{N}}$ be a binary sequence which is Cohen generic again with respect to M. Even though now we have the Cohen complete Boolean algebra C in M with respect to which $c = \{j_k\}_{j\in\mathbb{N}}$ is generic and C is not chosen from the lattice of projections and the indices are not connected with the operators and hence the lattice of projections. Still one can formulate the analogous result like for the measure algebra B

Corollary 2 *Let $\{a_{j_k}\}, k \in \mathbb{N}$ be a sequence of s.a. commuting operators on $\mathbb{C}^n$ and $\{j_k\}_{k\in\mathbb{N}}$ be a ZFC Cohen generic binary sequence of indices with respect to M. Then the tensor product $a_{j_1} \otimes a_{j_2} \otimes \cdots$ determine $a \oplus \mathcal{A}_c$ where a is a s.a. operator on $\mathcal{H}^{(\infty)}$ and $\mathcal{A}_c$ is certain s.a. operator on $\mathcal{H}^{\infty}$.*

This follows from Lemma 3 since $c = \{j_k\}_{j\in\mathbb{N}} \in 2^{\omega}_M \subset \mathbb{R}$ and $\mathbb{R} \hookrightarrow R_{V^B}$ and from Lemma 2 so one is choosing $\mathcal{A}_c$ as corresponding to $c \in R_{V^B}$. The first part of this Corollary refers to the of s.a. operator a which is the ITP limit of the corresponding finite tensor products completed (by the products of identities) (see ref. Landsman 2020).

From Corrolaries 1 and 2 it follows

Proposition 3 *There exist s.a. operators $\mathcal{A}^M_c, \mathcal{A}^M_r$ on $\mathcal{H}^{\infty}$, corresponding to Cohen and random generic reals with respect to any CTM M of ZFC.*

We are going to recognise properties of the sets of correlations $Corr(\mathcal{A}^M_r, \mathcal{A}^M_c)$ of the operators $\mathcal{A}^M_c, \mathcal{A}^M_r$ on $\mathcal{H}^{\infty}$ vs. on products $\mathcal{H}_a \otimes \mathcal{H}_b$ of Hilbert spaces. It holds

Theorem 5 *Let M be a CTM of ZFC, then*

a) $[\mathcal{A}^M_r, \mathcal{A}^M_c] = 0$.

b) There exist certain correlations $Corr(\mathcal{A}^M_r, \mathcal{A}^M_c)$ on $\mathcal{H}^{\infty}$ which can not be reproduced by any correlations $Corr(a \otimes \mathbb{1}, \mathbb{1} \otimes b)$ on $\mathcal{H}_a \otimes \mathcal{H}_b$ with a, b s.a. on any $\mathcal{H}_a$ and $\mathcal{H}_b$ correspondingly.

The proof goes as follows. First let us check the commutativity of $\mathcal{A}^M_r, \mathcal{A}^M_c$ in V. The random real $r \in R_{M[r]}$ and the Cohen real $c \in R_{M[c]}$ are also reals in $\mathbb{R}$ in V (Lemma 3). Then from Lemma 2 it follows that commuting s.a. operators on $\mathcal{H}^{\infty}$ in B in V are in 1:1 correspondence with reals R_{V^B} in V^B (see the SM A file) and $\mathbb{R} \hookrightarrow R_{V^B}$. From the constructions of both operators (Corollaries 1 and 2) it follows that they correspond to $r, c \in R_{V^B}$ so $[\mathcal{A}^M_r, \mathcal{A}^M_c] = 0$.

Next let us consider the set of correlations $Corr(\mathcal{A}^M_r, \mathcal{A}^M_c)$ on $\mathcal{H}^{\infty}$. It holds (relative to M)

$$\text{Given } \mathcal{A}^M_r \text{ the operator } \mathcal{A}^M_c \text{ is excluded;} \tag{1}$$

$$\text{Given } \mathcal{A}^M_c \text{ the operator } \mathcal{A}^M_r \text{ is excluded.} \tag{2}$$

This is due to Lemma 1 stating that given a random real r in $M[r]$ there can not exist any generic Cohen real c in $M[r]$ and conversely, so thus the operators corresponding to the generic reals inherit this property.

Let us consider now the set of correlations $Corr(a \otimes \mathbb{1}, \mathbb{1} \otimes b)$ on $\mathcal{H}_a \otimes \mathcal{H}_b$. The strong negative correlations as in (1) and (2) are not reproducible on the product $\mathcal{H}_a \otimes \mathcal{H}_b$. To see this let $c \in M[c]$ be Cohen generic real and $r \in M[r]$ a random generic real and $\mathcal{A}^M_r$, $\mathcal{A}^M_c$ be the corresponding s.a. operators on $\mathcal{H}_a$ and $\mathcal{H}_b$ correspondingly. Let us suppose that $\mathcal{H}_a = \mathcal{H}^{\infty}_a$ and $\mathcal{H}_b = \mathcal{H}^{\infty}_b$. This supposition is not any limitation since taking two finite dimensional Hilbert spaces their product already can not reproduce all correlations on $\mathcal{H}^{\infty}$. To show that the exclusions (1), (2) do not hold on the product $\mathcal{H}_a \otimes \mathcal{H}_b$ let us take $\mathcal{A}_r^M \otimes \mathbb{1}$ and $\mathbb{1} \otimes \mathcal{A}_c^M$ on $\mathcal{H}_a \otimes \mathcal{H}_b$. One sees that

$$\text{Given } \mathcal{A}_r^M \otimes \mathbb{1} \text{ the operator } \mathbb{1} \otimes \mathcal{A}_c^M \text{ is not excluded;} \tag{3}$$

$$\text{Given } \mathcal{A}_c^M \otimes \mathbb{1} \text{ the operator } \mathbb{1} \otimes \mathcal{A}_r^M \text{ is not excluded.} \tag{4}$$

The reason for this non-exclusions is that now we have independence of Hilbert spaces in the product which extends over independence of generic extensions and the corresponding operators which means (in the first case of $\mathcal{H}^\infty$) $M[r]$ excludes $M[c]$ and conversely, while in the case of $\mathcal{H}_a \otimes \mathcal{H}_b$ the extensions appear freely. This finishes the prove of Theorem 5.

To complete the proof of Theorem 4 one can not deal directly with the $\mathcal{A}_r$ and $\mathcal{A}_c$ operators and their correlations since they are presumably not finitely decomposable like $\mathcal{A} = \sum_{i=1}^N \lambda_i P_i, \lambda_i \in \mathbb{R}, i = 1, 2, \cdots, N$. However, the existence of $\mathcal{A}_r$ and $\mathcal{A}_c$ indicates on the existence of the extended universes $M[r]$ and $M[c]$ in which one can find generic finitely decomposable operators $\mathcal{A}^{M[r]} = \sum_{i=1}^N \lambda_i^{(r)} P_i^{(r)}, \lambda_i^{(r)} \in R_{M[r]}, i = 1, 2, \cdots, N$ for its finitely many eigenvalues and spaces in $\mathcal{H}^{M[r]}$ in the extended model $M[r]$ such that some of its eigenvalues are M B-generic. Similarly for certain s.a. operator $\mathcal{A}^{M[r]}$ in $M[c]$ such that $\mathcal{A}^{M[c]} = \sum_{i=1}^N \lambda_i^{(rc} P_i^{(c)}, \lambda_i^{(c)} \in R_{M[c]}, i = 1, 2, \cdots, N$ which some of its eigenvalues are M C-generic. It should also hold $\sum_{i=1}^N P_i^{(r)} = Id$ on $\mathcal{H}^\infty$ in $M[r]$ and $\sum_{i=1}^N P_i^{(c)} = Id$ on $\mathcal{H}^\infty$ in $M[c]$ and $[P_i^{(r)}, P_j^{(c)}] = 0$ for all $i, j = 1, 2, \cdots, N$. How do we know such generics exist? Let us start with certain s.a. operators $\mathcal{P}$, $\mathcal{Q}$ on $\mathcal{H}^\infty$ in V with the properties we need

$$\mathcal{P} = \sum_{i=1}^N \lambda_i P_i, \lambda_i \in \mathbb{R}, i = 1, 2, \cdots, N$$

$$\mathcal{Q} = \sum_{i=1}^N \kappa_i Q_i, \kappa_i \in \mathbb{R}, i = 1, 2, \cdots, N$$

$$\sum_{i=1}^N P_i = \sum_{i=1}^N Q_i = Id \text{ on } \mathcal{H}^\infty \text{ and } [P_i, Q_j] = 0 \text{ for all } i, j = 1, 2, \cdots, N \text{ (in } V) .$$

Projections P_i, Q_j as well $\mathcal{H}^\infty$ can be defined in $M[r]$ and $M[c]$ Benioff 1976a. For example given the projection say

$$P_i = \text{ 'the projection on } i\text{th 1-dimensional complex subspace in } \mathcal{H}$$

and this can be directly interpreted in $M, M[c], M[r]$ just remembering that the complex numbers describing the subspace are pairs of reals from $R_M, R_{M[r]}, R_{M[c]}$ the subsets of $\mathbb{R}$. Real numbers (eigenvalues) $\lambda_i, \kappa_j, i, j = 1, 2 \cdots, N$ are taken in such a way that at least one λ_i (at least one κ_j) are M B-generic in $M[r]$ (M C-generic in $M[c]$). This choice requires eventual tiny modifications of real coefficients since for the Cohen generic extension $M[r]$ the generic reals have Lebesgue measure 1 (full) in $R_{M[c]}$ and the random generic have outer measure 1 in $R_{M[r]}$. Now let us take $\mathcal{A}^{M[r]} = \mathcal{P}$ and $\mathcal{A}^{M[c]} = \mathcal{Q}$ which have the desired properties.

Note that working in V we do not face the obstructions like in (1), (2) since the nontrivial generics do not exist in V. The central question seems to be: What is the QM reason that experiments alignments and QM are shrunk and collapse to the models like $M[r], M[c]$? The answer is *genericity* of QM on $\mathcal{H}^\infty$ (see Subsections 4.2 and 4.5 and Lemma 6). What is more, when observables $\mathcal{P}$ and $\mathcal{Q}$ commute in V (i.e. the corresponding projections commute) they are simultaneously measured.

Thus given 'QM is ZFC random' there exists a random generic $r \in M[r]$ of outcomes of QM and QM is in $M[r]$. We will show that the strong negative correlation (1) vs. uncorrelated case (3) apply for $\mathcal{A}^{M[r]} = \mathcal{P}$ in $M[r]$ and $\mathcal{A}^{M[c]} = \mathcal{Q}$ in $M[c]$ as above.

Let us describe the usual physical settings for TC N. Ozawa 2013. On $\mathcal{H}^\infty$ Alice (a) is performing measurements over the system S and she has a finite set of positive bounded operators $\mathcal{A}_{i\alpha} \in \mathbb{A}_{\{i,\alpha\}} \subset \mathcal{L}(\mathcal{H}^\infty)$ where i is the index of the ith operator and α indicates certain outcome resulting in the measurement of operators from $\mathbb{A}_{\{i,\alpha\}}$. There are finitely many of the all possible outcomes αs. Similarly Bob (b) is performing independent measurements over S with $\mathcal{H}^\infty$ and he has a finite $\mathbb{B}_{\{j,\beta\}}$ set of positive bounded operators indexed by j with the finite set of all possible outcomes βs. Under the assumption that all $\mathcal{A}_{i\alpha} \in \mathbb{A}_{\{i,\alpha\}}$ commute with all $\mathcal{B}_{j\beta} \in \mathbb{B}_{\{j,\beta\}}$ (independence of measurements by a and b) and given a state ω – a positive linear normalised functional $\omega : \mathcal{L}(\mathcal{H}^\infty) \to \mathbb{R}$, one can represent the set of possible correlations arising in the measurements of S by a and b on $\mathcal{H}^\infty$

$$Corr(\mathcal{A} \cdot \mathcal{B}) = \{prob(i,j|\alpha,\beta) = \omega(\mathcal{A}_{i\alpha} \cdot \mathcal{B}_{j\beta}) \text{ on } \mathcal{H}^\infty\}_{ij\alpha\beta}. \tag{5}$$

On $\mathcal{H}_a \otimes \mathcal{H}_b$ the independence of measurements by a and b is guaranteed by the separation between the corresponding Hilbert spaces of S on which a and b are performing their measurements. The state ω is now the functional $\omega : \mathcal{L}(\mathcal{H}_a \otimes \mathcal{H}_b) \to \mathbb{R}$, $\mathbb{A}_{\{i,\alpha\}} \subset \mathcal{L}(\mathcal{H}_a)$, $\mathbb{B}_{\{j,\beta\}} \subset \mathcal{L}(\mathcal{H}_b)$ which results in the set of correlations

$$Corr(\mathcal{A} \otimes \mathcal{B}) = \{prob(i,j|\alpha,\beta) = \omega(\mathcal{A}_{i\alpha} \otimes \mathcal{B}_{j\beta}) \text{ on } \mathcal{H}_a \otimes \mathcal{H}_b\}_{ij\alpha\beta}. \tag{6}$$

The TC is the hypothesis that for any settings as above $Corr(\mathcal{A} \cdot \mathcal{B}) = Corr(\mathcal{A} \otimes \mathcal{B})$. Let us apply the generic QM into such formulated TC. Working in V both a and b are performing their experiments in the settings above and both B-genericity (QM is ZFC random) and C-genericity are available. In the measurement on $\mathcal{H}^\infty$ the set universe V (context) is collapsed, $V \to M$, such that the random extension after the measurement $M[r]$ is nontrivial (see SM B). Then according to Lemma 6 the reduced operator $\mathcal{A} \to \mathcal{A}^{M[r]}$ is measured. Similarly to Bob's measurement $\mathcal{B}$ is collapsed to $\mathcal{B}^{M[c]}$ and measured in $M[c]$. Thus there is the possibility that $\mathcal{A} \in \mathbb{A}_{\{i,\alpha\}}$ and the set of αs includes eigenvalues of $\mathcal{A}$ (finite number) and $\mathcal{B} \in \mathbb{B}_{\{j,\beta\}}$ and the eigenvalues of $\mathcal{B}$ are in the set of βs. Then let $\lambda_i \in R_{M[r]}$ be one of the generic eigenvalues of $\mathcal{A}^{M[r]}$ in $M[r]$ and κ_j be generic of $\mathcal{B} = \mathcal{A}^{M[c]}$ in $M[c]$. Then the strong negative correlation of (1) applies to $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{H}^\infty$ for the projections corresponding to these generic eigenvalues.

Still with $\mathcal{A} = \mathcal{A}^{M[r]}$ and $\mathcal{B} = \mathcal{A}^{M[c]}$ let the Hilbert space be the product $\mathcal{H}_a \otimes \mathcal{H}_b$ and the operators result as $\mathcal{A}_a^{M[r]} \otimes \mathbb{1}$ and $\mathbb{1} \otimes A_b^{M[c]}$. They do not face the negative correlations above and hence follow the pattern of (3) and (4) and, since there are no obstructions coming from single models and generics the negative correlations do not emerge and are not reproducible here. This completes the proof of Theorem 4.

Remark 7 *One could wonder whether the asymmetry in assigning the random extensions to a and Cohen extension to b is valid. In fact it can be performed conversely; it is also possible that $M[r]$ is assigned to both a and b or that $M[c]$ is assigned to the both and this does not change the result. In fact one should follow the rule of equal accessibility of the choices by both participants. Otherwise there would be dependence of choice b on the choice a which we want to exclude. Still the choice in the proof is possible and when followed leads to the result; see SM C file for this generalised standpoint.*

Remark 8 *One could also wonder whether genericity is indeed required for the result of Theorem 4. Let us assume that c and r were non-generic and they exist already in M as reals from R_M so they still exist in V^B and correspond to commuting s.a. operators. But then the exclusions* (1) *and* (2) *can not hold since Lemma 1 does not hold for non-generic reals.*

Thus in this approach the form of amplified randomness of binary sequences related with the ZFC genericity is indeed needed for negating the Tsirelson's conjecture.

The presented above results touch a fundamental issue in set theory – the existence of generic ultrafilters in general, however, it seems that the issue is somehow decisive in understanding randomness in QM on infinite dimensional Hilbert spaces, e.g. ref. Król, Bielas, and Asselmeyer-Maluga 2023. Genericity becomes a part of formalism of QM especially measurement process. This will be more thoroughly discussed in the next section (see also SM E). The QM contexts based on ZFC are models M, $M[G]$ and the like which become the 'worlds of classical discourse', or classical contexts, and come up here as bearing the nontrivial formal structure. The contexts, however, are not only representing the classical world of physics but influence also the quantum picture. This is particularly seen in the non simultaneity of contexts $M[r]$ and $M[c]$ which could be seen as yet another quantum like property. This particular point will be addressed separately.

4. Discussion and Supplementary Material

In this explanatory long section we present deepened discussion of various fundamental, though more distant to Theorem 4, aspects of QM formalism. Nevertheless, the Reader can find also here, as subsection 4.1, the proof of Proposition 2 missing in the main body of the paper.

In mathematics, the search for an answer to the question of whether generic filters (in particular over V) exist puts the problem in the spotlight alongside other ontological questions – and hence, in a sense, outside of the formalism of set theory Hamkins 2012. The existence of generic ultrafilters is not universal among models of ZFC, as was already emphasized earlier (see the SM A file). In particular, they do not exist for V and always exist for a CTM M. One solution is to refer to Boolean models like V^B – then generic ultrafilters will exist with the Boolean value 1. From the forcing point of view, referring to Boolean models in V is closely related to construing CTMs in V. Such an approach fixes V as an absolute environment that, in particular, cannot be extended: for instance, adding new reals to $R_V = \mathbb{R}$ will be pointless, as $\mathbb{R}$ already contains all reals, just as V should contain 'all' sets. However, the absolute character of the universe of sets (cumulative hierarchy of sets) can be relativized in the foundations of set theory, leading to the multiverse-based (MV-based) foundations. One of the main problems leading to MV is the relation to the existence of generic filters and extensions such as $V[G]$. As Hamkins explains, the point Hamkins 2012 generic filters G do not exist in V, but this should be viewed analogously to the nonexistence of $\sqrt{-1}$ in $\mathbb{R}$, and the extensions $V[G]$ would then correspond (only by historical similarity rather than by any actual correspondence) to $\mathbb{C}$. The Forcing Extension Principle, which is one of the axioms of the MV approach, states that *for any universe V and any forcing notion P in V, there will be a forcing extension $V[G]$, where $G \subset P$ is V-generic.* What strikes us as strange is that there seems to be yet another principle of MV in this direction, i.e. Absorption into **L** (the constructible Gödel universe), where this says that every universe V will be a countable transitive model in another universe W that satisfies $V = \mathbf{L}$. Not only is V in W countable, but also W can be chosen as a constructible universe. However, a usual feature of this approach is that V is typically a non-well-founded (nonstandard) model as seen by W. Thus, MV places us in a realm where countability and well-foundedness are mutually relative aspects, and there will not be any absolute or distinguished universe V fixing the meaning of countability and well-foundedness. These two radically different approaches in the foundations of set theory, one based on universe V and the other on multiverse MV, now belong to the scope of QM considerations.

We seem to be at a rather peculiar point, where experiments can shed light on an abstract but crucial decision in the foundations of set theory: namely, whether or not generic filters in V exist, and whether the V-based or MV-based approach is more likely to prevail over the other one. The approach to TC presented here assumes the ZFC genericity of QM and the existence of ZFC random sequences of outcomes, but nevertheless it may happen to be verified experimentally (along with the distinction TC and $\neg$TC). In Subsection 4.5, we have shown that ZFC genericity is a part of the standard QM formalism when completed by the set theory part.

Three scenarios are therefore possible: the first points to M and its generic extension $M[G]$

in which QM is formulated and M would be physically real – i.e. with V excluded due to the $\neg$(V-genericity) property. This would mean that set theory, construed as underlying our physical reality, is shrunk down to model M, but universe V is still a suitable point of reference for describing M, and V-based theoretical foundations would thus prevail over MV ones. In classical physics, or for non-extreme energy scales, this distinction could prove irrelevant. If it happens that experiments really show that QM is ZFC generic, then the V-based formalism of classical physics will not be extendable over QM, which would have to be formulated in a ZFC model M supporting the existence of generic filters – or, alternatively, the paradigm would have to be radically changed. This second paradigm points to the MV-based foundations of set theory, where genericity is assured at every stage, where this would dominate in the quantum regime (on $\mathcal{H}^{\infty}$).

There is also a third possibility, which seems most natural from the point of view of the methods adopted in this paper: namely, that true universes are local Boolean-valued models V^B. This would not contradict the V-based approach, realizes genericity, and is closed to the CTMs, as well as following directly from the QM formalism. (See the discussion in the Subsection 4.5). Moreover, this last option seems to be well-suited for addressing an issue such as the overlapping domain of space-time with a quantum regime – something that will be a topic of a separate work from the present authors. (However, see ref. Król and Asselmeyer-Maluga 2020).

There is a certain formal context which already favors the MV approach in QM and is related to hidden variables of QM. This is the mutual genericity of QM. The appearance of genericity in QM has already been analysed by Robert van Wesep (Van Wesep 2006), William Boos (Boos 1996) and Paul Benioff (Benioff 1976a, 1976b), and their arguments referred to in ref. Król, Bielas, and Asselmeyer-Maluga 2023 (as well as in a series of papers by the present authors and their collaborators – see, e.g., Król and Asselmeyer-Maluga 2022; Król 2004, 2016; Król and Klimasara 2020). In the context of quantum physics there is also a work by Illias Farah and Menachem Magidor (Farah and Magidor 2012) showing the independence of Pitovsky's construction of spin chains on ZFC axioms. The method applied is forcing, and some large cardinal supposition. Still, not many papers so far have focused on the issue of genericity in QM. The message coming from the works by Wesep and Boos is that if Solovay genericity were to be removed from the QM formalism, this would contradict the nonexistence of hidden variables in their strong form in QM. That means, in particular, that QM is formulated in a formal ZFC environment which supports mutual genericity: if it were V at some stage then, formally, there would not be any generic filter, and hence no genericity would be possible. It should be MV or V^Bs, as $V^B \subset V$ reintroduces genericity in V. (See the supplementary subsection 4.5.) Genericity has also been present in QM – albeit in an implicit, finite form – in a work by Paterek et al. 2010. The authors in question have shown that logical independence, and hence forcing-related construction, is a strong principle in QM, such that it leads to quantum phenomena. This work by Paterek et al. has dealt with finite-dimensional Hilbert spaces, but based on our present work we are able to discern two main directions for extending such an approach. One is the logical independence of finite axiomatic systems over infinite axiomatic ones. The important observation of ref. Paterek et al. 2010 is the replacement, for the Heisenberg group of observables, of quantum randomness by purely classical logical independence. The partial (though maximal) information contained in finite sets of axioms/states concerning experimentally verifiable formulas/sentences is responsible for the inherent randomness of quantum outcomes. This goes in a reverse direction from the usual way of seeing such things. In refs. Król and Asselmeyer-Maluga 2020; Król, Bielas, and Asselmeyer-Maluga 2023, randomness in QM for infinite-dimensional $\mathcal{H}$s has been represented by forcing genericity in models of ZFC, where this points to logical independence on ZFC axioms. ZFC cannot be finitely axiomatized, so such a ZFC-based approach in QM indicates that the infinite-dimensional $\mathcal{H}$ should be related to independence in formal systems having an infinite number of axioms. This, and the finite case of independent sentences, point to the peculiar fact that logical independence may underlie randomness in QM. The other extension of the work in ref. Paterek et al. 2010 comes from quantum-resource theory. (The present authors wish to thank Roberto Salazar for drawing their attention to QRT.) In ref. Król, Bielas, and Asselmeyer-Maluga 2023 we have encountered the possibility of QM being formulated in a model M of ZFC, rather than in the entire universe V of sets. A pinpointing of a certain QRT for the 'QM in *M*' situation might facilitate further pursuit of an eventual experimental verification. These two scenarios are currently subject to investigation, and will be addressed separately.

We have presented an approach in which genericity and forcing (and hence logical independence on ZFC) may lie at the heart of QM on infinite dimensional Hilbert spaces and can allow us, in principle, to distinguish TC and ¬TC. Still, the important issue is whether one could distinguish TC and ¬TC experimentally, and how to make use of this eventually over-correlated phenomenon in practice. One natural expectation would be that we could design and build a new generation of quantum random-number generators certified by ¬TC correlations. As abstract as this looks now, it must nevertheless await future endeavors such as will allow, in particular, a grasp of ¬TC correlations. Certainly, additional - also theoretical - work is needed. This is related to the Solovay randomness or ZFC randomness of QM, which has been assumed here when proving Theorem 4, but the formalism of QM on infinite-dimensional Hilbert spaces is closely related to this Król, Bielas, and Asselmeyer-Maluga 2023 (see the SM E). A related problem, then, is whether the ZFC randomness of QM outcomes can be turned into real new generators. It seems at present that a way towards a proper understanding of the state of the art of ¬TC correlations could go through their relation to the no-signaling theories exploring extensions of standard QM. Again, work on this is currently in progress.

This confirmation of ¬TC continues to stand in a certain analogy with the other problem raised by Tsirelson: i.e. whether the infinite dimension of a Hilbert space can be seen in a finite number of bipartite correlations Tsirelson 1993. This has been resolved in the affirmative in ref. Coladangelo and Stark 2020, where the present authors have shown that five such correlations already serve to distinguish an infinite dimension of Hilbert spaces from a finite one. Thus, in principle, the infinite dimension represents a special and distinct realm of QM; on the other hand, performing real experiments to confirm it seems currently out of reach. The second problem of Tsirelson discussed here presents some further, probably more complex, refinement of the infinite-dimensional case of QM.

The supposition in Theorem 4 that QM is ZFC-random therefore calls for comment. The 'If' part will not be needed anymore in the case discussed above, where there are experimentally verified ZFC-generic sequences in QM, or in the case where ¬TC requires necessarily (referring to) such generic sequences as are not known at present. So far it has been shown that one reason for ¬TC is the genericity, or ZFC randomness, of QM. We have shown in the Subsection 4.5, that QM can already be viewed as ZFC-random, with some elaboration of the set-theory component of QM that is directly determined by the structure of $\mathbb{L}$. However, whether this possibility is indeed adhered to by quantum phenomena must await future verification. From the purely formal point of view, when the set-theory component is represented, QM can be seen as ZFC-random on infinite-dimensional Hilbert spaces.

The use of ZFC models as structural components of physical theories has already proven to be a valid tool, in the sense that ZFC models serve as *physical degrees of freedom* of a kind. In particular, such an approach helps us to understand the regime in which space-time becomes a quantum object Król and Asselmeyer-Maluga 2022, 2020. On the other hand, formal methods based on models of set theory, even though entirely assigned to the domain of ZFC, might well come to be rediscovered and found to be highly useful right at the heart of QM considerations.

4.1 The proof of Proposition 2

Let $\sigma = \{i_k\}_{k\in\mathbb{N}} \in 2^\omega$ be a ZFC random binary sequence of indices relative to a transitive standard ZFC model M.

First we choose a model M as CTM of ZFC. Let $\mathcal{H}^{(\infty)}$ be a separable infinite dimensional complex Hilbert space and $\mathbb{L}(0, 1, \wedge, \vee) = \mathbb{L}$ be its corresponding lattice of projections. The maximal Boolean algebra of projections $\mathcal{B}$ chosen from $\mathbb{L}$ reads Groote 2005; Kadison and Ringrose 1997

$$\mathcal{B} = B_a \oplus B$$

where B_a is certain atomic Boolean algebra corresponding to the finite dimensional, while B is the complete atomless Boolean algebra which is always present for the infinite dimensional Hilbert spaces (and it is absent for the finite dimensions). Moreover, this B is always isomorphic to the measure algebra $Bor(\mathbb{R})/Null$ of Borel subsets of $\mathbb{R}$ modulo the ideal of the subsets of the Lebesgue measure zero.

Let $B_M = B$ be the measure algebra relative to M, i.e. we use the same symbol B for the measure algebra outside the model M (in V) and in M. In M one defines the Boolean-valued universe of sets M^B, however, being construed in M it allows for the canonical embedding $M \hookrightarrow M^B$. M^B is not any 2-valued universe of sets but it is rather the universe with B-valued logic such that the theorems of 2-valued ZFC are formulas with logical value 1. To retrieve the 2-valued model one performs the quotient construction by a generic filter $\mathcal{U}$ in B (in M). The result is the generic 2-valued extension of the model M

$$M^B/\mathcal{U} \simeq M[\mathcal{U}].$$

This $M[\mathcal{U}]$ is precisely the random forcing extension of M. It is marked as $M[r]$ where r is a random real adjoined to M. Usually there are plenty of such generic random reals which come along with each random real r. The point is that for a CTM M of ZFC there always exists a generic filter $\mathcal{U}$ as above so does a generic real. Moreover, whenever B is a complete atomless in M then there always exists the nontrivial forcing extension $M \subsetneq M[r]$ as above. In general, however, such generic filter may not exist.

Now let us turn to $M = V$ the von Neumanns cumulative universe of sets. Here there emerges the problem with non-existence of any generic filter for B in V. The solution is again referring to the construction of Boolean-valued universe of sets V^B in V. Then one proves Viale, Audrito, and Steila 2019, p.70

$$V^B \vdash [\![\exists_{\mathcal{U}}\mathcal{U} \text{ is a generic filter in } B]\!] = 1 \in B \tag{7}$$

which allows to overcome the difficulty and consider the Boolean universe V^B as necessary step in the construction. Then the usual construction for the random forcing extension is valid and reads (to be sure that $V^B/\mathcal{U}$ is standard, one should refer to ω-ZFC models, i.e. with the standard natural numbers object).

$$V^B/\mathcal{U} \simeq V[r], \text{ where } r \text{ is a random generic real (relative to } B \text{ in } V). \tag{8}$$

One such random real represents the binary sequence $\sigma = \{i_k\}_{k\in\mathbb{N}} \in 2^\omega$ of indices as in the Proposition 2 in the main text. The Remark 4 in Results section leads to the conclusion that given a set of Hilbert spaces, say $\mathbb{C}^2_{i_k}$, $k \in \mathbb{N}$, it is a random sequence of the spaces. But this gives rise to the splitting of the randomness of the indices from the ITP construction performed on spaces. But the first is a random real r and the ITP is a nonseparable $\mathcal{H}^{(\infty)}$. It remains to show that r determines a s.a. operator on a separable H^∞. This follows from general properties of Boolean valued models of ZFC in the context of the QM lattice of projections $\mathbb{L}(\mathcal{H}^\infty)$. V^B is the model of ZFC which means it has an object of real numbers $R^{(B)}$, however, the logic of V^B is Boolean and not 2-valued and the object of reals is non-trivially bigger than real numbers $\mathbb{R}$ in V. This has been analysed already by Takeuti 1978 while developing the Boolean valued analysis. The equation (7) indicates the status of the generic real r as in equation (8). Namely

$$\exists_{r\in R^{(B)}} [\![r \text{ is } V\text{-generic random real}]\!] = 1 \in B.$$

Thus this is not any real $r \in \mathbb{R}$ added to V but rather a Boolean real $r \in V^B$. Now we can interpret this real as follows.

Lemma 4 (Takeuti 1978) *Real numbers in V^B are in 1:1 correspondence with the partitions of unity E in B.*

Remark 9 *We say that a s.a. operator $\mathcal{A}$ is in B when all the projections from defining it spectral family $\{E_\lambda\}$, are in B.*

Lemma 5 (Takeuti 1978) *The partitions of unity E in B are in 1:1 correspondence with the s.a. operators in B on $\mathcal{H}^\infty$.*

This last Lemma follows from the spectral decomposition of any s.a. operator $\mathcal{A}$, i.e. $\mathcal{A} = \int_{sp(\mathcal{A})} \lambda E_\lambda$, $\forall_\lambda E_\lambda \in B$, where $E = \{E_\lambda\}$ is the spectral family of $\mathcal{A}$ and it is the partition of unity in V^B at the same time.

Taking ultrafilter quotient as in (8) we find that in $V[r]$ the Boolean generic real corresponds to certain s.a. operator $\mathcal{A}$. Thus we conclude

Corollary 3 *In V there exists the s.a. operator $\mathcal{A}$ on $\mathcal{H}^\infty$ corresponding to the V-generic Boolean r in $V[r]$.*

And the statement as in Proposition 2 follows: $\bigotimes_{i_k}^\infty \mathbb{C}^2_{i_k}$ *determines the ITP $\mathcal{H}^{(\infty)}$ and certain s.a. operator $\mathcal{A}$ on a separable infinite dimensional Hilbert space $\mathcal{H}^\infty$.*

4.2 Measurements in QM and genericity

We extend the measurement rule in V formulated in Król, Bielas, and Asselmeyer-Maluga 2023, Secs. 2.1.2; 3.4.1. In V one has both B and C algebras so thus one can build the sequences which *become* generic over V^B and V^C the Boolean-valued models of ZFC (recall that both algebras, measure and Cohen, are atomless complete Boolean algebras). In V such sequences can coexist since they are not B nor C V-generic. Hence the elavation $V \to V^B \to V^B/\mathcal{U}$ or $V \to M$ leads directly to B- or C-genericity: $V^B/\mathcal{U} \simeq V[\mathcal{U}]$ or $M \to M[\mathcal{U}]$ (same goes for C).

> [QM Generic Measurement] Generic QM measurements on $\mathcal{H}^\infty$ are paired with the random forcing extensions $V \to V[\mathcal{U}]$ or $M \to M[\mathcal{U}]$. The universe V is thus changed to the relative universe V allowing for the nontrivial forcing extensions $V[\mathcal{U}]$. Equivalently V can be seen as replaced by a CTM M in V allowing for the nontrivial $M[\mathcal{U}]$ as well.

We use the same symbol M for the V which allows for nontrivial forcing extensions (e.g. in MV approach) and for M - a CTM replacing V. So V is the cummulative universe of sets not allowing directly for generic extensions. It follows that

Lemma 6 *In a generic measurement on $\mathcal{H}^\infty$ certain C-sequences in V (non-generic) become C-generic in M.*

This follows from the absoluteness of the Cohen algebra for CTMs and thus the construction of generic filters in V in $M \subset V$ become C-generic relative to M. Given a CTM M there always exist C-generic reals and $R_M \subset R_{M[c]} \subset \mathbb{R}$.

4.3 Generic extensions and $\neg$TC

In general there are four ways how the assignments of two generics into the two sets of operators $\mathbb{A}$ (of Alice) and $\mathbb{B}$ (of Bob) can be done. On $\mathcal{H}^\infty$ this results in the following possibilities

$$\begin{aligned}
\text{Given } \mathcal{A}^{M[r]} \text{ then } \mathcal{B}^{M[c]} \text{ is excluded;}\\
\text{Given } \mathcal{A}^{M[r]} \text{ then } \mathcal{B}^{M[r]} \text{ is not excluded;}\\
\text{Given } \mathcal{A}^{M[c]} \text{ then } \mathcal{B}^{M[r]} \text{ is excluded;}\\
\text{Given } \mathcal{A}^{M[c]} \text{ then } \mathcal{B}^{M[c]} \text{ is not excluded.}
\end{aligned}$$

While on the product space $\mathcal{H}_a \otimes \mathcal{H}_b$ the corresponding possibilities read

$$\begin{aligned}
\text{Given } \mathcal{A}^{M[r]} \text{ then } \mathcal{B}^{M[c]} \text{ is not excluded;}\\
\text{Given } \mathcal{A}^{M[r]} \text{ then } \mathcal{B}^{M[r]} \text{ is not excluded;}\\
\text{Given } \mathcal{A}^{M[c]} \text{ then } \mathcal{B}^{M[r]} \text{ is not excluded;}\\
\text{Given } \mathcal{A}^{M[c]} \text{ then } \mathcal{B}^{M[c]} \text{ is not excluded.}
\end{aligned}$$

These cases repeat the part of the argument in the proof of Theorem 4 in the broadest context.

4.4 ITPs and sequences of QM measurements

Here we follow the original proof in ref. Landsman 2020 of 1-randomness and the construction of the Born's probability on infinite sequences of QM outcomes. This analysis contains the ITPs of Von Neumann and the explicit use of quantum states which were missing in the main body of the paper. This construction deals with the measurement procedure on ITP and the infinite sequences of measurements.

Let H be a finite dimensional complex Hilbert space and $B(H)$ the algebra of all bounded operators on $\mathcal{H}$. Then $a = a^*, a \in B((H)$ means a is self-adjoint. First it is considered the following set of commuting operators $a_1, \cdots, a_N$ on the finite tensor product Hilbert space $\mathcal{H} = \bigotimes_{i=1}^N H_i$ where all $H_i = H, i = 1, \cdots, N$ (we can take $H_i = \mathbb{C}^2, i = 1, 2, \cdots, N$)

$$[a_1 = a \otimes 1_H \otimes \cdots \otimes 1_H]; \cdots; [a_N = 1_H \otimes \cdots 1_H \otimes a]. \tag{9}$$

Having fixed such elementary setup we can extend this up to infinite products of Hilbert spaces and consider Born's probability on the resulting space. Still we follow ref. Landsman 2020. The operator $a \in B(H)$ determines the maximal commutative C^*-algebra, $C^*(a)$, which is isomorphic to the algebra of continuous functions on the spectrum $s(a)$, $C^*(a) \simeq C(s(a))$. In the finite dimensional case $s(a)$ contains the eigenvalues of a, i.e. $\lambda_a \in s(a), a\psi = \lambda_a\psi$, $\psi \in H$. For the family (9) of commuting operators, a, there is the joint spectrum $S(\mathsf{a})$ and the C^*-algebra generated by the operators, $C^*(\mathsf{a})$ which still has its isomorphic representation, $C^*(\mathsf{a}) \simeq C(S(\mathsf{a}))$ where now $S(\mathsf{a})$ contains all nonzero eigenvalues $\lambda(\mathsf{a}) = (\lambda_1, \cdots, \lambda_N)$ for the same joint eigenvector $\psi \in \mathcal{H}$, i.e. $a_1\psi = \lambda_1\psi, \cdots, a_N\psi = \lambda_N\psi$. Let $e_{\lambda_i}, i = 1, \cdots, N$ be the projections of a_i on the one-dimensional subspaces corresponding to λ_i in H and ω_{a} a state on $B(\mathcal{H})$ determining the Born's join probability of finding $\lambda(\mathsf{a})$ in a measurement, which reads (provided $e_{\lambda(\mathsf{a})}$ is the projection of a on $e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_N} \neq 0$)

$$p(\lambda(\mathsf{a})) = \omega(e_{\lambda(\mathsf{a})}).$$

For the finite dimensional case this reduces to the probability determined by a density operator ρ, i.e. $p(\lambda(\mathsf{a})) = \mathrm{Tr}(\rho(e_{\lambda(\mathsf{a})}))$. Representing the density operator corresponding to a pure state as $\rho = |\psi\rangle\langle\psi|$

for a unit vector $\psi \in \mathcal{H}$, we obtain the Born's formula for the joint probability

$$p(\lambda(\mathbf{a})) = \langle\psi|\,(e_{\lambda(\mathbf{a})})\psi\rangle.$$

Since ω is a state on $B(\mathcal{H})$ and the measurement of $\mathbf{a}$ is performed on the (system with) N-tensor product space in the state ω such that this is generated by the N-fold measurements of $a \in B(H)$ on H, as in (9), in the state $\omega_1 \in B(H)^*$. It should hold

$$\omega = \omega_1^N \text{ which acts on the } N\text{-fold tensor product of vectors by}$$

$$\omega_1^N(e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_N}) = \omega_1(e_{\lambda_1}) \cdots \omega_1(e_{\lambda_N}) = p(\lambda_1) \cdots p(\lambda_N).$$

This is the joint probability in the measurement for the entire run of the finite sequence of N measurements. Next let us extend the procedure over the infinite sequences. The strict relation between the Born's probability of a single measurement of a and the Born's probability of the whole-run of the infinitely many measurements as in $\mathbf{a}$, holds true, provided the von Neumann complete ITPs are applied. Following ref. Landsman 2020 there are crucial regularity properties of the Nth fold tensor product of the spectral C^*-algebras

$$1)\ B(H)^{\otimes N} = B(H\otimes\cdots\otimes H);\quad 2)\ C^*(a)^{\otimes N} = C^*(a_1, \cdots, a_N),\ \text{for } a \in B(H), a_1, \cdots, a_N \in B(\mathcal{H} = H^{\otimes N}).$$

One way to make sure that the limiting cases: $\lim_{N\to\infty} H^{\otimes N}$ and $\lim_{N\to\infty} B(H)^{\otimes N}$ are the complete ITP H^∞ of von Neumann and $B(H^\infty)$, correspondingly, is by taking the class of infinite sequences $\mathbf{a} = (a_1, a_2, \cdots)$ such that for any such sequence there exists $M \in \mathbb{N}$ and $a_M \in C^*(a)^{\otimes M}$ such that for any $N > M$, $a_N = a_M \otimes 1_{C^*(a)} \otimes \cdots \otimes 1_{C^*(a)}$. Based on this condition one shows that

$$\lim_{N\to\infty} H^{\otimes N} = H^\infty \text{ the complete ITP, and}$$

$$\lim_{N\to\infty} B(H)^{\otimes N} = B(H^\infty), \text{ the bounded operators on the complete ITP } H^\infty,$$

$$\lim_{N\to\infty} C^*(a)^N = C^*(a)^\infty = C(s(a))^{\mathbb{N}} = C^*(a_1, a_2, \cdots).$$

This last case corresponds to the infinite product of compact measure spaces (note that here we have $\mathbb{N}$ - the set of all natural numbers). Also in this infinite case given the state ω_1 on $B(H)$ one uniquely determines the state ω_1^∞ on $B(H)^{\otimes\infty} = B(H^\infty)$. Finally, the measure $\mu_{\mathbf{a}}$ derived from the infinitely long sequences of operators, $\mathbf{a}$, is determined by the one element sequence, a (see (9)), i.e.

$$\mu_{\mathbf{a}} = \mu_a^\infty.$$

There are other possible constructions of ITPs, like grounded tensor product of countably many grounded Hilbert spaces Baez, Segal, and Zhou 1992 or inductive limits of the inductive family of the finite subsets of indices set Guichardet 1992. These both examples are more physics directed since the resulting ITPs are already separable Hilbert spaces. However, they can be also realized as closely related to the truncated von Neumann ITP by taking suitable set of infinite sequences $\mathbf{a}$.

4.5 ZFC twist of QM

This supplementary section explains the relation of the standard QM with QM where infinite sequences of outcomes are explicitly Solovay random. The ZFC randomness of infinite sequences is, in fact, formally present already in the Born's probability and this presence becomes transparent under additional specification of the set theory side of QM. This is rather a change of the point of view to the complementary to quantum logic one, than the change of the standard QM formalism.

Below we will demonstrate briefly how the sets perspective emerges from logical one in QM on Hilbert spaces. Taking the set theory completion of QM directly refers to ZFC randomness. In theorem 4 in the main body of the paper we have assumed the ZFC randomness of QM and then found the way how to negate TC. This section shows that ZFC randomness is already a part of QM formalism when performing the set theory completion.

Solovay randomness requires infinite dimension of Hilbert space. Let $\dim \mathcal{H} = +\infty$ and $\mathbb{L}(\wedge, \vee, 0, 1)$ the lattice of projections on $\mathcal{H}$ then the maximal complete Boolean algebras chosen from it all have the isomorphic atomless part which is the measure algebra B. This was explained already at the beginning of the Subsection 4.1. Thus QM on infinite dimensional Hilbert space is always paired with a quantum logic carried by the lattice structure with the B as above. Our concern here is quantum set theory which is also determined by $\mathcal{H}^{(\infty)}, \mathbb{L}(\wedge, \vee, 0, 1)$ and B but this requires some additional clarification. First of all, is there any well-defined meaning assigned to something like *quantum set theory*. The answer can encompass the variety of proposals, though one is especially natural from the formal point of view. This follows the logic of QM on $\mathbb{L}$ and the way how Boolean-valued models of ZFC are construed. Let us start with 2-valued Boolean models of ZFC (where $2 = \{0, 1\}$ is 2-valued Boolean algebra). The connection of this 2-valued logic with models of ZFC is given trivially as $V \simeq V^2$ or $M \simeq M^2$, where V and M are class model and set model of ZFC, correspondingly. Thus 2-valued models of ZFC are (isomorphic images of) the models. The next step would be V^B (M^B) for arbitrary complete Boolean algebra B in V (M) replacing the algebra 2 above. Such Boolean-valued models of ZFC lie in the heart of Boolean-valued analysis which has been intensively developed since its invention by Gaisi Takeuti, Dana Scott and Robert Solovay in 1970ties. The models V^B have been also exploited in the main body of this paper where B was the complete atomless measure algebra $Bor(\mathbb{R})/\mathcal{N}$ with the relations between the models $V^B \hookrightarrow V \hookrightarrow V^B$. The next step is the full quantum set theory. Takeuti Takeuti 1978 proposed it to be the q-universe of sets, i.e. the lattice-valued model $V^{\mathbb{L}}$ in analogy with the Boolean-valued models V^B. However, $V^{\mathbb{L}}$ is very complicated and seemed to Takeuti untractable as a resonable universe of sets (though see the recent development by Ozawa M. Ozawa 2021). However, the family of Boolean-valued models $\{V^B\}$ is a good approximation of $V^{\mathbb{L}}$ as each V^B corresponds to the local Boolean contexts of quantum theory. On the other side the contexts are given by maximal families of commuting observables. The entire lattice model $V^{\mathbb{L}}$ can be engulfed, or covered, by Boolean models V^B but the family of such V^Bs has to be augmented by some class of relations between them. The precise structure of the relations is not relevant here and we can express the following correspondences on $\mathcal{H}^{(\infty)}$ between logic, sets and operators

$$\begin{aligned}
\{\text{logical local contexts of QM}\} &\longleftrightarrow \{\text{measure Boolean algebras, } Bs\}\\
\{\text{maximal algebras of commuting observables in QM}\} &\longleftrightarrow \{\text{measure Boolean algebras, } Bs\}\\
\{\text{set theoretic local contexts of QM}\} &\longleftrightarrow \{\text{Boolean-valued models of ZFC, } V^Bs\}\\
\text{global set theory of QM} &\longleftrightarrow \text{non-Boolean, non-Heyting universe of sets } V^{\mathbb{L}}.
\end{aligned}$$

This last follows from the observation that each maximal Boolean algebra $B \subset \mathbb{L}$ leads to $V^B \subset V$ which is a submodel of the quantum set theory universe $V^{\mathbb{L}}$. The second correspondence expresses the fact that for any family of commuting self-adjoint observables there always exists a maximal Boolean algebra B of projections determining all the operators (the spectral families of these self-adjoint operators have values in projections from B, e.g. ref. Takeuti 1978).

So far we have the universe $V^{\mathbb{L}}$ approximated by the family $\{V^B\}$ of Boolean-valued ZFC universes as local contexts. It remains to describe the step from Boolean many-valued to 2-valued contexts. This is also where Solovay randomness emerges. In general one refers to ZFC models M with the internal measure algebra B_M and builds in M the Boolean-valued model $M^{B_M} \subset M$ of ZFC. Let us use M^B for this M^{B_M} as was the case in the main body of the paper. This M^B is the carrier of the Solovay randomness via the forcing extension $M[r] \simeq M^B/Ult_r$ whre r is a Solovay random real. General results in set theory indicate that one has nontrivial forcing extension of the model M whenever the algebra B is complete and atomless. This is precisely the case for the measure algebra $Bor(\mathbb{R})/\mathcal{N}$ determined by the QM lattice $\mathbb{L}$. Thus the way from the non-distributive logic of QM and Boolean contexts V^Bs to the 2-valued world with 2-valued logic and set theory goes in infinite dimension, via random forcing extensions $V[r_\alpha]$, $\alpha \in I$, since these last models are already 2-valued and standard. This formal reasoning is a part of mathematics of QM and even though it bears the attribute of formal necessity, one can still ask whether QM as physical theory realizes such scenario. We already know that a single run of quantum measurement performed on a system with $\mathcal{H}_i$ has the property that its Born's measure determines the unique probability measure on the space of infinite repetitions on $\bigotimes_{i\in I} \mathcal{H}_i$, such that the 1-randomness of the sequences is maintained by the single run Born's rule. Does this single run on $\mathcal{H}_i$ determine the Solovay genericity of the outcomes?

In general one can construct the formalism of QM in models of ZFC, or in the universe V, this is also true for majority of mathematical constructions. QM is not any exception in this regard (though see ref. Benioff 1976a, 1976b where it has been shown that not all models are equally good). Moreover, any ZFC provable property is valid in every model of ZFC. Thus if the observer would use ZFC-provable techniques they remain the same in every universe of set and their richness does not matter since the models are indistinguishable (or simply there are no models). Set theory of QM indicates rather different behavior regarding the states: they might belong to different models though this is not observable, so the Born's probability is not affected. The variety of Boolean-valued universes is present in the QM formalism which follows the set theory perspective.

The following statement is the condition for the outcomes of QM measurements be ZFC random (Solovay generic). This is a reformulation of the generic measurement property discussed in Subection 4.2 and was already referred to, in a bit different context, in ref. Król, Bielas, and Asselmeyer-Maluga 2023

> *The Measurement Postulate of q set theory.* The Born measure of a single quantum run determines the Solovay genericity of infinite sequences of QM runs iff there exists a model M of ZFC such that QM measuring process of the quantum system admits description in M, i.e. the quantum state of the system before measurement is in M along with the preparing procedure for the mesurement, while after the measurement the quantum state of the system is in $M[r]$, where $r \in M[r]$ is a generic real and has the binary development representing the infinite sequence of the QM outcomes of measurements.

In the standard QM the set theory perspective is usually missing. The set theory perspective is rather the change of the point of view (the twist) in QM than the true change of the standard QM. Quantum set theory is complementary to quantum logic in the standard QM formalism.

Still remains unanswered whether the Solovay randomness (ZFC randomness in the terminology of ref. Król, Bielas, and Asselmeyer-Maluga 2023) can really be seen in experiments directly. This is quite analogous to the common use of infinite dimensional Hilbert spaces in QM while any discrimination of finite and infinite dimensional cases experimentally is not possible at present Coladangelo and Stark 2020.

Notes

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6. Mathematical Thinking for Sustainable Development

Małgorzata Gosek

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Mathematical Thinking for Sustainable Development

Małgorzata Gosek

Introduction

The Earth is on the verge of losing stable and human-friendly living conditions. Human activity, which is degrading the environment on a planetary scale and causing climate warming, is bringing about this.

For over 3 billion years, interactions between the geosphere and the biosphere controlled global environmental conditions. The state of the Earth's system changed due to forces caused by external disturbances or internal processes within the geosphere or biosphere. Today, however, there is a new force—human activity—and the anthroposphere has become an additional functional element of the Earth system, capable of altering its state (Richardson et al., 2023).

The climate and environmental crisis requires swift and effective action—transforming the economy and businesses, enacting wise legal regulations, and, most importantly, driving social change towards responsible consumption and pro-environmental attitudes. However, knowledge about the environmental crisis does not sufficiently permeate social awareness. While some improvement is visible, much must be done (Stefaniuk, 2021). People struggle to access accurate information amidst the flood of ecological fake news, and they have difficulty understanding how their daily choices and activities contribute to environmental degradation, as well as the consequences this brings for them and the planet. Environmental education, which is insufficient in schools, does not help (Gosek, 2023).

The implementation of the concept of sustainable development, which emerged as a remedy for the crisis and was intended to lead to economic and social changes, continues to face numerous obstacles despite the efforts of communities and states. The ease with which disruptions in action are justified by external factors, such as the COVID-19 pandemic, is noticeable (Council of Ministers..., 2023, pp. 13-14). It also seems that there is a lack of understanding of the interdependencies between the Sustainable Development Goals. Recognising these connections and strategically planning for them would enable better control over their implementation. However, this requires specific cognitive competencies that allow these relationships to be decoded.

This paper aims to demonstrate how mathematical and systems thinking can support the resolution of the environmental crisis and enhance the effectiveness of implementing sustainable development principles. The study employs an analytical-synthetic method based on literature and reports. The starting point is a synthetic overview of the contemporary climate and environmental crisis and the challenges in implementing sustainable development. In the next step, systems thinking and mathematical thinking are applied to understand the causes of the crisis and the barriers to sustainable development implementation. The analysis shows that effectively overcoming the crisis requires understanding the complexity of systems (natural and their connected systems—social and economic) and the intricate interactions between the Sustainable Development Goals. For this, systems thinking, supported by mathematical thinking, is essential. Developing these competencies is therefore crucial in education for sustainable development and is one of the conditions for successfully addressing the climate and environmental crisis.

The Contemporary Climate and Environmental Crisis

Anthropogenic pressure, mainly through the emission of greenhouse gases, has led to the warming of the atmosphere, oceans, and land. Between 2011 and 2020, Earth's surface temperature rose by 1.1°C above levels recorded in 1850-1900 (IPCC, 2023, p. 4). Future projections leave no illusions that this process will continue. In the five greenhouse gas emission scenarios analysed in the IPCC report, the year 2100 is expected to increase the global surface temperature from about 1.4°C to over 4°C above pre-industrial levels (IPCC, 2023, p. 7).

However, climate change is not the only concern. Equally troubling are the data on environmental degradation. Forty per cent of soils are severely degraded, and the loss of arable land occurs over 100 times faster than its natural formation (Hickel, 2022, p. 19). Agriculture alone is responsible for 80% of global deforestation, 70% of the decline in biodiversity in terrestrial ecosystems, 50% of the loss in freshwater biodiversity, 70% of freshwater use, and 29% of greenhouse gas emissions (Kramarz, 2022, p. 436). From agricultural lands, 89 million tons of synthetic fertilisers not absorbed by the soil flowed into water bodies out of the 200 million tons applied, destroying the environment. Additionally, 4 million tons of toxic pesticides are used annually to protect crops and orchards from weeds and pests that harm plants, insects, birds, and human health (Pomianek, 2023; Pomianek, 2024).

The ongoing changes are vividly illustrated by the concept of Planetary Boundaries, which view the planetary system as an integrated socio-ecological system consisting of nine processes responsible for the stability and resilience of our planet. All of these processes are critically affected by human activity. Recent data show that six of these processes have already exceeded safe boundaries. Besides climate change, they include biosphere integrity, novel environmental entities, biochemical flows, land system change, and freshwater use. Ocean acidification remains within the safe zone (though close to the limit), along with aerosol loading and stratospheric ozone levels (Richardson et al., 2023). The Planetary Boundaries model highlights that exceeding any individual boundary destabilises the entire system and underscores the importance of the feedback loops. However, it does not account for the socio-economic factors, processes, and structures driving the ecological crisis. Therefore, an alternative concept introduces the notion of social boundaries, defined by societies as self-imposed limits and conditions for “a good life for all” (Brand et al., 2021, pp. 265, 268).

The literature calls for humanising and socialising the discussion of the environmental crisis and drawing on expertise from fields like degrowth ecological economics, which offers concrete proposals for transformation (Hickel, 2021; Raworth, 2021; Bińczyk, 2023, pp. 14-

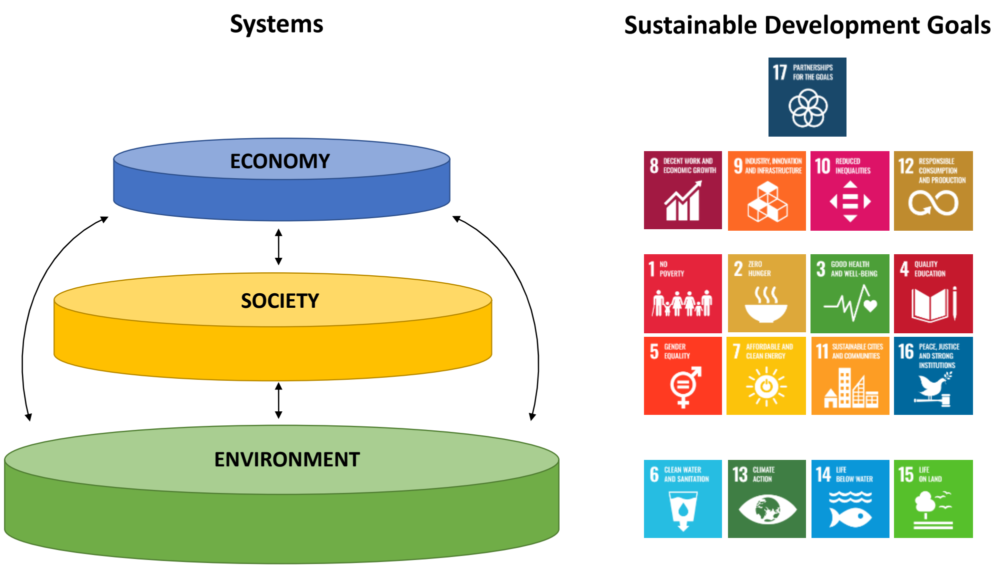
15). These approaches emphasise the need to recognise the interconnections between environmental, social, and economic spheres to understand the root causes of the crisis.

Sustainable Development as an Idea and Goal

Environmental problems and the need to solve them were recognised many years ago. In 1987, the World Commission on Environment and Development stated in its report: “At the current level of civilisation, sustainable development is possible, meaning development in which the needs of the present generation can be met without compromising the ability of future generations to meet their own needs.” Since then, sustainable development has become a subject of academic interest and an idea that has begun to be implemented.

In 2015, the United Nations (UN) adopted the 2030 Agenda, which includes 17 Sustainable Development Goals (SDGs) to be achieved by 2030 (www4). These goals relate to three systems: environment, society, and economy, outlining a vision of a healthy natural environment (Goals 6, 13, 14, and 15), a prosperous, equal, and inclusive society with access to quality education for all, living in health and peace (Goals 1, 2, 3, 4, 5, 7, 11, and 16), and an economy based on responsible consumption and production, sustainable industrialisation, stable infrastructure, with no inequalities between countries, and decent work for all (Goals 8, 9, 10, 12)—all of this supported through partnerships and cooperation (Goal 17) (Figure 1).

Figure 1. Sustainable Development Goals concerning the natural, social, and economic environments.



Source: Own work based on online sources (www3; www5).

Sustainable development has been incorporated into the Treaty on European Union and the 2001 EU Sustainable Development Strategy. In subsequent years, the 2030 Agenda goals were aligned with European strategies, European Commission priorities, and other documents. The European Green Deal (2019), the EU's new economic strategy, aims for the EU economy to be climate-neutral, resource-efficient, and circular by 2050 (www1; European Commission, 2019).

Global and EU strategies and regulations have been translated into national frameworks. In Poland, the response to the 2030 Agenda was the adoption of the Responsible Development Strategy (Ministry of Development, n.d.).

The implementation of the SDGs has also reached the level of businesses, which significantly impacts environmental and social issues. This shift is driven by enforcing regulations, pressure from conscious consumers, and growing awareness among corporate leaders.

The effectiveness of implementing the 17 SDGs is continuously monitored globally, at the European level, and within individual countries, with indicators made available to the public (Sachs et al., 2023; Council of Ministers..., 2023; www1; www2). Businesses' sustainability reporting (ESG reporting) is becoming increasingly widespread, although it is not yet mandatory in all countries. A notable example of mandatory reporting is in the EU, where in 2022, a directive was introduced requiring certain entities (with an expanding scope) to include information on environmental, social, and governance issues in their business reports, following common European sustainability reporting standards, known as ESRS (Directive of the European Parliament…, 2022). These regulations are driving profound changes in corporate thinking and operations. Moreover, an approach based on transparency throughout the value chain will also compel changes among suppliers and distributors (Redqueen, 2023, p. 33). These changes are becoming increasingly visible and will continue to expand.

Sustainable Development Goals – Interconnections and Consequences

In pursuing Sustainable Development Goals (SDGs), it is essential to recognise that they offer a holistic and multidimensional view of development and cannot be treated as standalone objectives. They must function as a system of interlocking gears and should not be seen as an additive structure but as a system of synergistic reinforcement (Pradhan et al., 2017, p. 1177). Only by achieving all goals can the global community realise the expected outcome.

However, this approach to the SDGs is not universally adopted. Some awareness of the interconnections between the goals exists among policymakers and implementing bodies. For instance, in a report on public administration activities in Poland, the connections between the main goal and other goals were identified (Council of Ministers..., 2023, pp. 148–156). The situation is less consistent in business operations, where goals are often pursued selectively without more profound reflection.

A lack of clearly defined relationships between goals and an insufficient understanding of these interconnections leads to incoherent policies during implementation (Costanza et al., 2016; Nilsson et al., 2016; Le Blanc, 2015). Analyses of SDG interactions and identified synergies (positive correlations between pairs of indicators) and trade-offs (negative correlations) reveal that the success of the SDG agenda will largely depend on the ability to harness synergies and address the trade-offs that obstruct goal achievement. Moreover, ensuring that these trade-offs do not become structural obstacles or, if necessary, implement more profound structural changes (Pradhan et al., 2017).

Ecosystem as a Complex System and Interconnected Systems

In addition to recognising the relationships between the SDGs, it is equally important to diagnose the interconnections within the ecosystem and the links between the ecosystem and the social and economic systems.

A natural ecosystem is a complex system consisting of many interconnected elements that interact non-linearly, leading to emergent properties and behaviours in the system as a whole.

Complex systems are characterised by non-linear behaviour stemming from long and feedback-driven chains of cause-and-effect relationships due to interactions among their elements. They have the capacity for self-organisation and evolutionary potential, dependent on the strength of internal interactions and interactions with the environment. In stable, unstable, and borderline states, complex systems can exhibit three qualitatively different types of behaviour: small changes or no changes, disorganised changes, and patterned behaviour that is unpredictable in detail. Complex systems demonstrate emergent behaviours—properties of the whole system that do not directly result from the behaviour of individual elements but are outcomes of the interactions and organisation of these elements. Some systems can adapt (Balcerak, 2020, pp. 6–7).

The ecosystem can be considered complex for several reasons. It includes numerous elements (plants, animals, fungi, bacteria, etc.) with diverse functions and roles. A network of interactions exists among species and organisms (e.g., food chains, symbiotic relationships). Actions and interactions between organisms lead to the emergence of new features and behaviours (e.g., self-regulation capacity). The ecosystem is dynamic and subject to changes due to shifting environmental factors (e.g., climate change). The processes occurring in the ecosystem are non-linear, and their outcomes are difficult to predict.

Understanding the ecosystem requires an interdisciplinary approach and analysis at different levels, from micro-scale (inter-organism interactions) to macro-scale (global processes).

A notable example illustrating the difficulty of understanding ecosystems is the Biosphere 2 project (1980s–90s), aimed at creating an alternative to Earth's biosphere. A complex was built in the U.S. over 12,700 square meters in size, including a tropical forest, mangroves, savanna, and ocean system. Eight people were supposed to survive there for two years without external contact, self-sufficiently producing drinking water, breathable air, and food. The project failed. Problems arose with food supply; all bees and hummingbirds went extinct, nematodes and mites attacked the plants, cockroaches multiplied, and the climate collapsed. The inhabitants had to rely on food supplies and oxygen from outside (Borejza, 2023). This failure starkly demonstrated how little humans know about the complexity of Earth's ecosystem.

In addition to understanding the ecosystem, it is equally important to grasp its connections with the social and economic systems. Climate change, through its impact on agriculture—the primary source of national income for underdeveloped countries—contributes to increased social inequalities. Global warming has caused a 25% increase in inequality between countries over the past half-century (Diffenbaugh & Burke, 2019). Climate warming also raises sea levels, increasing risks for many island nations, and causes droughts and limited access to freshwater (Budziszewska et al., 2021, p. 211). Consequently, it influences migration patterns. Between 2016 and 2021, more than 43 million children and young people in 44 countries worldwide were forced to leave their homes due to floods, droughts, or wildfires, and over the next 30 years, more than 100 million are expected to be displaced for similar reasons (Sieradzka, 2023). Regions expected to face food shortages and water-related problems include South Asia, Central and Southern Africa, and South America (Budziszewska et al., 2021, pp. 211–213). When paired with demographic forecasts that predict five of the ten most populous countries in the world in 2100 will be in Africa and four in Asia, the scale of migration becomes easy to imagine (Buchholz, 2020).

Climate warming also affects the economy. It contributes to the food crisis. A 1°C rise in global temperatures results in a 10% decline in crop yields, which, given current trends, will lead to a 30% reduction in global harvests within this century (Hickel, 2021, p. 29). It has caused growing economic losses due to extreme weather events ($264 billion in 2022) and heat exposure (490 billion lost potential working hours worldwide in 2022) (Karaczun, 2023).

Understanding the interconnections between the ecosystem and other systems is critical to effectively implementing sustainable development. However, it remains a significant challenge.

Systems Thinking

Systems thinking helps in understanding complex systems. This approach focuses on analysing and comprehending complex systems as a whole rather than merely as a collection of separate elements. Its main objective is to study the interrelationships, patterns, and processes within systems to understand better how they function and to predict how they will respond to changes. A systems perspective allows placing a specific event in a broader context, embedding it in time and space, defining the problem dynamically, and thus gaining deeper insights (Kronenberg & Bergier, 2010, p. 48).

The critical elements of a systems approach are the multitude of factors, mutual interconnections, and the closed chain of causes and effects, which form what are known as feedback loops. These can be reinforcing, driving system growth, causing rapid negative changes, or balancing and stabilising the system (Rokita, 2011, p. 90).

Systems thinking is beneficial not only for understanding ecosystems but also for implementing sustainable development principles. An example is the circular economy (CE)—an idea in which materials, components, and products (MCP) should be designed and produced to be restored, maintained, and redistributed within the economy for as long as possible in terms of environmental, technical, social, and economic factors (Hahladakis & Iacovidou, 2019). For successful implementation, it is essential to move away from a waste-focused system, where the quality of MCP degrades as they flow through production, consumption, and management systems, and to adopt a holistic approach that ensures solutions in one system (or system point) do not cause problems in another system (or another point of the same system). This is achieved through systems thinking, which examines subsystems' internal and external elements critical to transitioning to a circular economy and their interconnections. This approach allows for understanding how the resource recovery system evolves culturally, temporally, and spatially (Iacovidou et al., 2021, p. 24800).

Mathematical Thinking as a Complement to Systems Thinking

Systems thinking can be effectively supported by mathematical thinking, the ability to reason, analyze, and solve problems using mathematics. Mathematical thinking includes formulating logical arguments, thinking abstractly, and creating models and strategies for solving mathematical problems. Some of its manifestations include the ability to recognise and utilise analogies, schematisation and mathematisation, defining and interpreting definitions rationally, deduction and reduction, encoding, constructing, and using mathematical language rationally, as well as algorithmisation and the rational use of algorithms (Juskowiak & Mleczak, 2023, p. 73).

Mathematical and systems thinking share a similar approach to analysis, modelling, and problem-solving, and their integration can be highly effective. These approaches can complement and reinforce each other (Table 1).

Table 1. Systems Thinking and Mathematical Thinking – Complementary Approaches

Area	**Systems Thinking**	**Mathematical Thinking**
Problem-Solving	It helps identify general problems and interactions between system elements.	They are used to develop strategies for solving mathematical problems.
Understanding and Analyzing Complexity	Studies the relationships and interactions between system elements.	Provides tools to describe and analyse structures and processes.
Providing Abstract Models and Patterns	Provides systems thinking models (e.g., flow diagrams) to study interactions.	Provides mathematical structures (e.g., equations, functions, graphs) to describe phenomena and relationships.
Data Analysis and Drawing Conclusions	It helps identify patterns and trends in data and understand their causes.	Provides statistical, algebraic, and numerical techniques for data analysis.

Process Optimization and System Design	It helps design efficient, sustainable, and productive systems.	Provides tools to optimise functions and processes.

Source: Own work, partially based on the literature (Kronenberg & Bergier, 2010; Meadows, 2022).

Mathematical thinking thus aids in solving problems within complex systems. It can also help understand ecosystems, social and economic systems and their interactions.

The Need for Education for Sustainable Development

The need for a transformation in thinking about sustainable development has been identified as a crucial factor for successfully implementing the 2030 Agenda (Council of Ministers... 2023, p. 12). It is essential to understand the natural, social, and economic environments in which we live, the causes of the crisis, and the essence of the sustainable development concept being implemented so that actions taken can genuinely address the root causes of the problem. There is a growing need for a broader perspective, an understanding of system complexity, the ability to diagnose connections, and a critical view of reality. However, increasingly complex problems are emerging, while the ability for logical and analytical thinking, especially among the younger generation, is unfortunately not improving.

In ongoing discussions about education for sustainable development, the need to account for the complexity, ambiguity, and interdisciplinary nature of the problem of sustainable development has been emphasised. Education focused on multidisciplinarity, critical and systems thinking, and interdisciplinary skills have been proposed (Warburton, 2003). Suggestions have included using network science through peer-based, problem-oriented, and transformative approaches to learning (Weber, 2021). The need for holistic education that considers all aspects of human functioning in the socio-natural reality has also been highlighted (Mróz & Ocetkiewicz, 2019, p. 39). Proposals have been made to introduce an intermediate stage between the "how things are" knowledge phase and practical action – a stage of "broadening the mind," a kind of "mental gymnastics," which would include not only seemingly obvious things and phenomena but also methods of analysing phenomena based on comparisons, seeking relationships and analogies between seemingly distant events and facts (Górniak et al., 2003).

Thus, education for sustainable development should involve shaping mathematical and systems thinking. However, studies among teachers show that it is primarily associated with environmental protection. Systems thinking is not recognised at all as a critical competency in education for sustainable development, even though its development is a fundamental premise of the harmonious development concept (viewing the world as a complex system of interdependencies) (Mróz & Ocetkiewicz, 2019, p. 42).

Summary

Human activity has led to climate warming and environmental degradation on a planetary scale. Understanding the causes and mechanisms of these changes is a challenge for researchers and every individual. The adverse effects accelerate and reinforce each other when time runs out. Scientific knowledge is needed to design ways out of the crisis, and social awareness is required to change people's attitudes and pressure decision-makers, urging them to act. However, understanding the nature of the crisis requires delving into the complex structure of systems.

The same applies to attempts to implement sustainable development and achieve the 17 Goals. Despite determination, obstacles arise in implementation. This happens because the 17 Goals and their related 169 targets form one global objective for humanity, with many internal connections, the understanding of which is not simple. However, such understanding is necessary for success.

Understanding the complexity of systems is essential to effectively addressing the crisis, and for that, systems competencies and mathematical thinking are necessary.

Systems and mathematical thinking are pathways to understanding ongoing processes, the complexity of systems, and their interactions, as well as identifying the causes of climate and environmental crises. They are tools for planning and managing the implementation of Sustainable Development Goals and raising awareness of their interconnections. Systems and mathematical thinking should be mandatory education components for sustainable development, whether in school, university, or professional training.

However, rational arguments alone are not enough for success. Knowledge does not always lead to action. Emotions are needed – the experience of nature, a sense of connection, and the pursuit of harmony. There must also be a sense of responsibility for one’s children and grandchildren for future generations. Emotions and responsibility will translate into commitment and determination in action. There needs to be a return to values, transmitting them at home, building education around them in schools, and making them the core of business practices.

Without raising the level of education, society will not be able to push the political class to take necessary action, and systemic changes are essential if humanity is to begin emerging from the climate and environmental crisis.

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