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ARTICLE

Gravitation

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Abstract

Gravitation is introduced as a gauge field that couples to the impulsion in the Lagrangian approach to the classical mechanics. The equations of motion in a central static field yield an amazing result: a repulsive gravitational potential appears at short distance. As a consequence black holes would have a stable and finite size, thus eliminating gravitational singularities. The expansion of the universe and its acceleration could be explained without recourse to a hypothetical dark energy.

1. Introduction

A short historical review

After Einstein produced the famous General Relativity (GR) theory for gravitation, several theories were then also proposed to either improve, reinterpret or supplement GR. The aim of these efforts was to quantify gravitation and explain later discoveries like the expansion of the universe. Several of these theories are based on the non-zero torsion of the space time. Elie Cartan did develop a model where torsion would be generated by angular momentum Scholz 2018; Hehl and Obukhov 2007 Blagojevis and Hehl 2012: Cartan associates to each closed infinitesimal contour a rotation (which expresses curvature) and a translation (which expresses torsion). In Cartan's mind the rotation can be represented by a vector and the translation by a torque. Tetrad formalism, teleparallel gravity, Weitzenböck and Möller theories are shown to be equivalent to GR in reference Arcos and J G Pereira 2005, and reference Schucking 2008 shows that the Schwartzschild metric can have an interpretation of teleparallelism in the Pound-Rebka experiment. An Introduction to teleparallel gravity is given in reference Lurie 2013 where curvature =0 and torsion is the gravitational field strength. In consequence, there are no geodesics in Teleparallel Gravity, only force equations quite analogous to the Lorentz force equation of electrodynamics. The authors expected this result by because, like electrodynamics, Teleparallel Gravity is also a gauge theory. Gauge theories for gravitation are treated in Hayashi and Nakano 1967 in a formal mathematical frame, and in Arcos and J G Pereira 2005; Kleinert 2010 where a mathematical formalism in which torsion and curvature can be exchanged via a supergauge symmetry leads to the GR equations. Translation gauge potentials Ivanenko and Sardanashvily 1987 meet Cartan's idea of the spin of matter being the source of torsion: The gauge gravitation theory based on the relativity and equivalence principles reformulated in fibre bundle terms is the gravitation theory with torsion whose source is the spin of matter. "Since translation gauge potentials fail to be utilized for describing a gravitational field, a question on their physical meaning arises." "Therefore, translation gauge potentials may be responsible for weaker forces than gravity as discussed by some of the authors." Goldstonic supergravity, supergroups, supertransformations, superspace, supersymmetries, superfunctions etc. are examined in a mathematical frame in this paper. The expansion of the universe and its acceleration are explained in the framework of GR by the action of a phantom dark energy Peebles and Ratra 2002; Dutta and Scherrer 2009 in the ΛCDM model.

The present work

This is a new approach on gravitation. In the opposition to GR it is defined as a gauge theory with the gauge group of translations. The source is the energy impulsion tensor that is conserved which results from the invariance of the Lagrangian under the group of translations. Gauge theories introduced by Hermann Weyl postdate GR and are now a successful mathematical formalism in providing a unified framework to describe the quantum field theories of electromagnetism, the weak force and the strong force. Here our gauge symmetry group of translations is abelian as is the U(1) group for electromagnetism and this makes the mathematical treatment simpler. The graviton is defined as a massless spin 2 boson. At this stage one can notice that abelian gauge theories have massless gauge bosons with the infinite range.

We will follow the classical Lagrangian approach where space-time is distorted by the presence of a gravitational field.This distortion will induce a change into the metric. At infinite distance from massive bodies where the gravitational field tends to zero, the metric is Minkovskian $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. We will begin with the introduction of the gravitational potential in space-time and into the metric. With the Pound-Rebka effect as a first result. Then we will introduce the gravitational potential into the Lagrangian that leads to the classical equations of motion. This will lead to the equations of motion of a body in a gravitational field. The action change under gauge transforms of the gravitational field is evaluated, as well as the symmetry properties of the gravitational potential. The equations of the field are thus derived. We can then apply the equations of motion in a central static field, beginning with the deflection of light by a massive body, the sun in this case. Newton's law is then demonstrated, as is the principle of equivalence. The non-relativistic and relativistic equations of motion in a central static field are expressed in polar coordinates. The precession of Mercury's perihelion is calculated which fits well to the measured value. The equations also show that black holes must have a finite size, when the attractive and repulsive gravitational forces are in equilibrium. Some considerations on black hole properties follow, and also on the accelerated expansion of the Universe.

2. The gravitational field Γ^{μ}_{ν}

Time-space, vectors, metric, scalars and so on

In our 4-dimensional space-time, we can meet vectors (ex: x^{μ}), covectors (ex: p_{μ}), scalars that are the contraction of a vector on a covector: $x^{\mu}a_{\mu} = (x|a)$. A scalar can also be produced from two vectors x^{μ} and y^{ν} with the help of the metric $g_{\mu\nu}$: $x^{\mu}g_{\mu\nu}y^{\nu}$ and conversely $x_{\mu}g^{\mu\nu}y_{\nu}$ with two covectors x_{μ} and y_{ν} , $g^{\mu\nu}$ being the inverse matrix of $g_{\mu\nu}$. We can consider that $x_{\mu}g^{\mu\nu}$ is a vector x^{ν} that contracts with y_{ν} In the free (free means that there is no gravitation) space-time, the metric is $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ the Minkovskian metric where $\eta_{\mu\nu} = \eta^{\mu\nu}$ is its own inverse.

Now a word about the contraction operator that creates a scalar from a vector and a covector: Let the vector be $x^{\mu} = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \end{pmatrix}$ and the covector be $a_{\nu} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \end{pmatrix}$. The scalar $(x|a)$ is made by multiplying x^0 with a_0, x^1 with a_1 and so on and then taking the sum. It could have been $x^0a_1 + x^1a_2 + x^2a_3 + x^3a_0$ or any other mix of indices but it is not. The operator governing this contraction is the Kronecker δ^{ν}_{μ} that will make that x^0 matches with a_0, x^1 with a_1 and so on. Thus, we write: $(x|a) = x^{\mu}\delta^{\nu}_{\mu}a_{\nu}$ Also $x_{\mu}g^{\mu\nu}y_{\nu} = x_{\mu}\delta^{\mu}_{\lambda}g^{\lambda\rho}\delta^{\nu}_{\rho}y_{\nu}$: here we have a double contraction on μ and ν. All this may seem trivial, but we will demonstrate that the gravitational field acts on the contraction operator δ^{ν}_{μ} in the following way: δ^{ν}_{μ} becomes $\delta^{\nu}_{\mu} + \Gamma^{\nu}_{\mu}$ when acting on the impulsion

p_ν. For instance, this modifies the differential of the action

$$dS = p_\mu dx^\mu \text{ to } dS' = p_\mu(\delta^\mu_\nu + \Gamma^\mu_\nu)dx^\nu = p_\mu dx^\mu + p_\mu \Gamma^\mu_\nu dx^\nu$$

. This can be compared to the action that includes an electromagnetic field A_μ: $dS' = p_\mu dx^\mu + eA_\mu dx^\mu$ Mass m will also be a subject to the change in a gravitational field. One had $m^2 = p_\mu \eta_{\mu\nu} p_\nu$ or $m^2 = \frac{E^2}{c^4} - p^2$ in the free space. In a gravitational field $p_\mu = \delta^\nu_\mu p_\nu$ becomes

$$p_\mu = (\delta^\nu_\mu + \Gamma^\nu_\mu)p_\nu \text{ and } m'^2 = p_\lambda(\delta^\lambda_\mu + \Gamma^\lambda_\mu)\eta^{\mu\nu}(\delta^\rho_\nu + \Gamma^\rho_\nu)p_\rho \simeq m^2 + 2p_\mu \Gamma^\rho_\nu \eta^{\rho\nu} p_\nu + \Gamma\Gamma \text{ terms.}$$

This explains the slight mass gain of a body moving in a central static potential. We can also write: $m^2 = p_\lambda g^{\lambda\rho} p_\rho$ with the metric in a gravitational field becoming $g^{\lambda\rho} = (\delta^\lambda_\mu + \Gamma^\lambda_\mu)\eta^{\mu\nu}(\delta^\rho_\nu + \Gamma^\rho_\nu)$ a metric for covectors. And $g_{\lambda\rho} = (g^{\lambda\rho})^{-1} = (\delta^\mu_\lambda - \Gamma^\mu_\lambda)\eta_{\mu\nu}(\delta^\nu_\rho - \Gamma^\nu_\rho)$ would be the metric for vectors. Developing to the first order in Γ we get: $g^{\lambda\rho} = \eta^{\lambda\rho} + 2\Gamma^\lambda_\mu \eta^{\mu\rho}$ and $g_{\lambda\rho} = \eta_{\lambda\rho} - 2\Gamma^\mu_\lambda \eta_{\mu\rho}$

It will be shown later that Γ^μ_ν is symmetrical in μ and ν and can be made diagonal by a suitable change of coordinates. This change leaves δ^μ_ν invariant. Then $g^{\lambda\rho}$ can be written, to the first order in Γ:

$$g^{\lambda\rho} = \begin{pmatrix} 1 + 2\Gamma^0_0 & 0 & 0 & 0 \\ 0 & -1 - 2\Gamma^1_1 & 0 & 0 \\ 0 & 0 & -1 - 2\Gamma^2_2 & 0 \\ 0 & 0 & 0 & -1 - 2\Gamma^3_3 \end{pmatrix}$$

And

$$g_{\lambda\rho} = \begin{pmatrix} 1 - 2\Gamma^0_0 & 0 & 0 & 0 \\ 0 & -1 + 2\Gamma^1_1 & 0 & 0 \\ 0 & 0 & -1 + 2\Gamma^2_2 & 0 \\ 0 & 0 & 0 & -1 + 2\Gamma^3_3 \end{pmatrix}$$

The Pound and Rebka experiment

The Pound and Rebka experiment shows how the time is modified by the gravitation: thus only Γ^0_0 is acting and dt^2 becomes $dt^2(1 - 2\Gamma^0_0)$. The scalar time lapse $|dt|$ becomes $|dt|(1 - \Gamma^0_0)$. Let us write Γ for Γ^0_0. The Figure 1 shows the layout of the experiment:

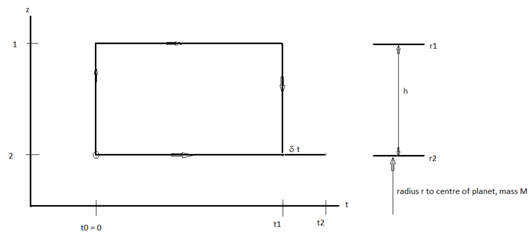

Figure 1. Pound-Rebka experiment

At point 1, $dt_1 = (1 - \Gamma_1)dt$. At point 2, $dt_2 = (1 - \Gamma_2)dt$, so that:

$$\frac{dt_1}{dt_2} = \frac{1 - \Gamma_1}{1 - \Gamma_2} \simeq 1 - \Gamma_1 + \Gamma_2 \tag{1}$$

$\frac{dt_1}{dt_2}$ was measured by Pound and Rebka in 1960 to be equal to: $1 - \frac{GM}{c^2}(\frac{1}{r_1} - \frac{1}{r_2})$. Which allows us to identify Γ^0_0 to $\frac{GM}{rc^2}$. Here M is the earth mass, G the gravitational constant, c the speed of light and $r_{1,2}$ the distance of points 1 and 2 to the center of Earth.

3. Equations of motion in the gravitational field Γ^{μ}_{ν}

Classical mechanics tells us that the trajectory of a body between two points A and B is the path where the integral of the action dS is minimal, with $dS = p_{\mu}dx^{\mu}$ and p_{μ} being the impulsion of the body. The occurrence of a field Γ^{μ}_{ν} will influence dx^{μ} and generate the equations of motion of a body in a gravitational field.

Let us consider the Lagrangian equation of motion, seen from a geometrical point of view. A small deviation from the trajectory between the two points A and B will not change the action to the first order:

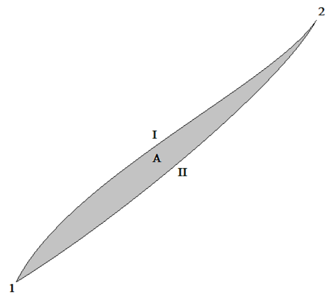

Figure 2. Action variation

$$\int_{I} p_{\mu}\, dx^{\mu} = \int_{II} p_{\mu}\, dx^{\mu}$$

or: $\oint_{I,-II} p_{\mu}\, dx^{\mu} = 0$ which is equivalent to: $\iint_{A}(\partial_{\mu}p_{\nu} - \partial_{\nu}p_{\mu})dS = 0$. Thus, $\partial_{\mu}p_{\nu} - \partial_{\nu}p_{\mu} = 0$

Multiplying by $v^{\mu} = \frac{dx^{\mu}}{dt}$, one gets: $\frac{dx^{\mu}}{dt}\frac{\partial p_{\nu}}{\partial x^{\mu}} - v^{\mu}\partial_{\nu}p_{\mu} = 0$. Since $\partial_{\nu}v^{\mu} = 0$ and $\frac{dx^{\mu}}{dt}\frac{\partial p_{\nu}}{\partial x^{\mu}} = \frac{d}{dt}p_{\nu}$ we get: $\frac{d}{dt}p_{\nu} - \frac{\partial}{\partial x^{\nu}}(v^{\mu}p_{\mu}) = 0$

But $v^{\mu}p_{\mu}$ is the Lagrangian $\mathcal{L}$, defined as the time derivative of the action:

$$S = \int p_{\mu}\, dx^{\mu} = \int p_{\mu}v^{\mu}\, dt = \int \mathcal{L}\, dt.$$

We can write the equation of motion as:

$$\frac{d}{dt}p_{\nu} - \frac{\partial}{\partial x^{\nu}}\mathcal{L} = 0$$

or $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v^{\nu}} - \frac{\partial}{\partial x^{\nu}}\mathcal{L} = 0$ with $p_{\nu} = \frac{\partial \mathcal{L}}{\partial v^{\nu}}$

When the gravitational field Γ^{μ}_{ν} is introduced we get the following action: $S = \int p_{\mu}(\delta^{\mu}_{\nu} + \Gamma^{\mu}_{\nu})\, dx^{\nu}$. Thus $S = \int p^{\mu}(\delta^{\nu}_{\mu} + \Gamma^{\nu}_{\mu})v^{\nu}\, dt = \int \mathcal{L}\, dt$ which implies

$$\mathcal{L} = p_{\mu}(\delta^{\mu}_{\nu} + \Gamma^{\mu}_{\nu})v^{\nu}.$$

Equations of motion

1st term:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v^{\nu}} = \frac{d}{dt}(p_{\nu} + \Gamma^{\mu}_{\nu}p_{\mu}) = \frac{d}{dt}p_{\nu} + p_{\mu}\frac{d}{dt}\Gamma^{\mu}_{\nu} + \Gamma^{\mu}_{\nu}\frac{d}{dt}p_{\mu} \tag{2}$$

$$= \frac{d}{dt}p_{\nu} + p_{\mu}v^{\rho}\frac{\partial \Gamma^{\mu}_{\nu}}{\partial x^{\rho}} + \Gamma^{\mu}_{\nu}v^{\rho}\frac{\partial p_{\mu}}{\partial x^{\rho}} \tag{3}$$

2d term:

$$\frac{\partial \mathcal{L}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}(p_\rho v^\rho) + \frac{\partial}{\partial x^\nu}(p_\mu \Gamma^\mu_\rho v^\rho) \tag{4}$$

We simplify the notation by: $\partial_\nu \equiv \frac{\partial}{\partial x^\nu}$. Then, since $\partial_\nu v^\rho = 0$

$$\partial_\nu \mathcal{L} = v^\rho \partial_\nu p_\rho + p_\mu v^\rho \partial_\nu \Gamma^\mu_\rho + v^\rho \Gamma^\mu_\rho \partial_\nu p_\mu \tag{5}$$

and assembling the simplified terms in the Lagrange equation, we get:

$$v^\mu \partial_\mu p_\nu + \Gamma^\mu_\nu v^\rho \partial_\rho p_\mu = p_\mu v^\rho (\partial_\nu \Gamma^\mu_\rho - \partial_\rho \Gamma^\mu_\nu) + v^\rho \Gamma^\mu_\rho \partial_\nu p_\mu + v^\mu \partial_\nu p_\mu \tag{6}$$

Or:

$$v^\mu(\partial_\mu p_\nu + \partial_\nu p_\mu) + v^\rho(\Gamma^\mu_\nu \partial_\rho p_\mu + \Gamma^\mu_\rho \partial_\nu p_\mu) = p_\mu v^\rho(\partial_\nu \Gamma^\mu_\rho - \partial_\rho \Gamma^\mu_\nu) \tag{7}$$

and finally

$$\begin{aligned} v^\mu(\partial_\mu p_\nu - \partial_\nu p_\mu) &= v^\rho[\partial_\nu(p_\mu \Gamma^\mu_\rho) - \partial_\rho(p_\mu \Gamma^\mu_\nu)] \\ \Leftrightarrow v^\rho(\partial_\rho p_\nu - \partial_\nu p_\rho) &= v^\rho[\partial_\nu(p_\mu \Gamma^\mu_\rho) - \partial_\rho(p_\mu \Gamma^\mu_\nu)]. \end{aligned} \tag{8}$$

The equation (8) is satisfied if

$$\partial_\rho(p_\nu + p_\mu \Gamma^\mu_\nu) - \partial_\nu(p_\rho + p_\mu \Gamma^\mu_\rho) = 0. \tag{9}$$

Temporal and spatial fields

Let us define $\gamma_\nu \equiv p_\mu \Gamma^\mu_\nu$ then, the equation (8) can be rewritten as:

$$v^\rho \partial_\rho p_\nu - v^\rho \partial_\nu p_\rho = v^\rho(\partial_\nu \gamma_\rho - \partial_\rho \gamma_\nu). \tag{10}$$

Or, with $v^\rho \partial_\rho = \frac{d}{dt}$ and $v^\rho \partial_\nu p_\rho = \partial_\nu v^\rho p_\rho = \partial_\nu \mathcal{L}$ as:

$$\frac{d}{dt} p_\nu = \frac{\partial \mathcal{L}}{\partial x^\nu} + v^\rho(\partial_\nu \gamma_\rho - \partial_\rho \gamma_\nu). \tag{11}$$

If no other extra fields are present, $\frac{\partial \mathcal{L}}{\partial x^\nu}$ can be ignored and for $\nu = 0$ one gets:

$$\frac{d}{dt} p_0 = v^i(\partial_t \gamma_i - \partial_i \gamma_0) \; ; \; i = 1, 2, 3. \tag{12}$$

Or, if we define $E_i \equiv \partial_i \gamma_0 - \partial_t \gamma_i$ we get:

$$\frac{d}{dt} p_0 = -v^i E_i. \tag{13}$$

For $\nu = 1, 2, 3$ noted as $\nu = i$:

$$\frac{d}{dt} p_i = v^\rho(\partial_i \gamma_\rho - \partial_\rho \gamma_i) \; ; \; \rho = 0, 1, 2, 3. \tag{14}$$

For $\rho = 0$, $v^0 \equiv 1$ and we get: $1(\partial_i \gamma_0 - \partial_t \gamma_i) = E_i$ again. For $\rho = 1, 2, 3$ noted as $\rho = j$ one has:

$$v^j(\partial_i \gamma_j - \partial_j \gamma_i) = \vec{v} \times \overrightarrow{rot\vec{\gamma}}. \tag{15}$$

If we define $\vec{H} \equiv \overrightarrow{rot\vec{\gamma}}$ we finally get:

$$\frac{d}{dt} \vec{p} = \vec{E} + (\vec{v} \times \vec{H}). \tag{16}$$

4. Symmetry of Γ_μ^ν

Γ_μ^ν is a mixed co and contravariant tensor (its product with $p_\nu v^\mu$ contributes to the Lagrangian as a scalar). This 4×4 tensor can have a symmetric and antisymmetric parts. We show that its antisymmetric part would correspond to a transform where the 4-length is preserved and there is no distortion.

If $dx^\mu dx^\nu \eta_{\mu\nu} = dx^2$ is the 4-length of the vector dx, then after Γ operation the length of the vector will become:

$$(dx^\mu + \Gamma_\lambda^\mu dx^\lambda)(dx^\nu + \Gamma_\rho^\nu dx^\rho)\eta_{\mu\nu} =$$
$$= dx^\mu dx^\nu \eta_{\mu\nu} + dx^\mu \eta_{\mu\nu}\Gamma_\rho^\nu dx^\rho + dx^\nu \eta_{\mu\nu}\Gamma_\lambda^\mu dx^\lambda + \eta_{\mu\nu}\Gamma_\lambda^\mu \Gamma_\rho^\nu dx^\lambda dx^\rho$$

To the first order in Γ, and if the length is preserved, we would get: $dx^\mu dx^\nu \eta_{\mu\nu} = dx^\mu dx^\nu \eta_{\mu\nu} + dx^\nu \Gamma_{\nu\rho} dx^\rho + dx^\mu \Gamma_{\mu\rho} dx^\rho$. Switching the dummy index ν to μ in the second term $2\Gamma_{\mu\rho} dx^\rho dx^\mu = 0$ and by symmetry of $dx^\rho dx^\mu$, $\Gamma_{\mu\rho}$ should be antisymmetric. The symmetrical part of Γ_μ^ν is $\eta_{\mu\lambda}\Gamma^{\nu\lambda}$ and also $\eta_{\mu\lambda}$ and $\Gamma^{\nu\lambda}$ both are symmetrical. Thus $\Gamma_\nu^\mu = \Gamma_\mu^\nu$ The symmetry of μ and ν in Γ_μ^ν ensures that it will produce no rotation (or more generally conservation of the 4-length) but it will produce only the distortion of space time. As a consequence Γ_μ^ν will have 10 components and Γ_μ^ν could possibly correspond to a spin 2 graviton. Feynman 1995

To make it visual, consider the following Figures 3 and 4. On Fig. 3, the symmetrical $\Gamma_1^2 = \Gamma_2^1$ creates a distortion in the 1, 2 plane. $\vec{1'} + \vec{2'} = \vec{1} + \vec{2}\Gamma_1^2 + \vec{2} + \vec{1}\Gamma_2^1 = (\vec{1} + \vec{2})(1 + \Gamma_1^2)$

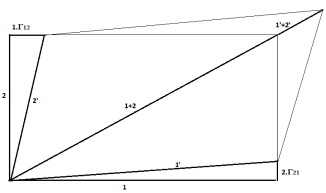

Figure 3. distortion

If $\Gamma_1^2 = -\Gamma_2^1$ then we would have a rotation instead of distortion, and as a consequence Γ_0^0 would also be = 0 instead of $\frac{GM}{rc^2}$. $\vec{1'} + \vec{2'} = \vec{1} + \vec{2}\Gamma_1^2 + \vec{2} + \vec{1}\Gamma_2^1 = \vec{1} + \vec{2}\Gamma_1^2 + \vec{2} - \vec{1}\Gamma_1^2$.

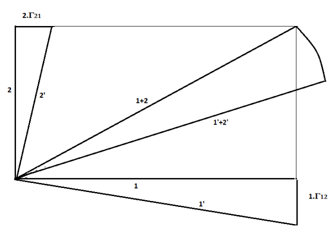

Figure 4. rotation

5. The equations of field

We look for an action that is a scalar, gauge invariant and that includes only the Γ_μ^ν terms. Such an action will be noted as S_f. Let us start from 11: $\frac{d}{dt}p_\nu = \frac{\partial \mathcal{L}}{\partial x^\nu} + v^\rho(\partial_\nu \gamma_\rho - \partial_\rho \gamma_\nu)$. The term

$(\partial_\nu \gamma_\rho - \partial_\rho \gamma_\nu)$ is invariant under gauge transform $\Gamma^\mu_\nu \to \Gamma^\mu_\nu + \frac{\partial_\nu \Phi}{p_\mu}$. The action term $\int p_\mu \Gamma^\mu_\nu \, dx^\nu$ is also invariant under the same gauge transform.

Expanding and neglecting $\frac{\partial \mathcal{L}}{\partial x^\nu}$ (no other potential but gravitational is present), we get:

$$\frac{d}{dt} p_\nu = v^\rho p_\mu \underbrace{(\partial_\nu \Gamma^\mu_\rho - \partial_\rho \Gamma^\mu_\nu)}_{I} + v^\rho \underbrace{(\Gamma^\mu_\rho \partial_\nu p_\mu - \Gamma^\mu_\nu \partial_\rho p_\mu)}_{II} \tag{17}$$

The term I is invariant under the gauge transform $\Gamma^\mu_\nu \to \Gamma^\mu_\nu + \partial_\nu \, G^\mu$. The gauge transform leaving invariant γ_ν and I must satisfy $\frac{\partial_\nu \Phi}{p_\mu} = \partial_\nu \, G^\mu$. The scalar term is $(\partial_\nu \Gamma^\mu_\rho - \partial_\rho \Gamma^\mu_\nu)^2$ and it is defined as:

$$(\partial_\nu \Gamma^\mu_\rho - \partial_\rho \Gamma^\mu_\nu) g^{\nu\alpha} g^{\rho\beta} g_{\mu\gamma} (\partial_\alpha \Gamma^\gamma_\beta - \partial_\beta \Gamma^\gamma_\alpha). \tag{18}$$

We define the action of the field as:

$$S_f = \alpha \int (\partial_\nu \Gamma^\mu_\rho - \partial_\rho \Gamma^\mu_\nu)^2 \, d\Omega. \tag{19}$$

With α an arbitrary constant, $d\Omega = \sqrt{-g}\, dVdt$, V is the spatial volume and $g = det(g_{ij}) = -1 + 2Tr\Gamma$, Γ is the Γ^μ_ν matrix. The action is now completed with the field – matter interaction term:

$$S = \int p_\mu \Gamma^\mu_\nu v^\nu \, d\Omega + \alpha \int (\partial_\nu \Gamma^\mu_\rho - \partial_\rho \Gamma^\mu_\nu)^2 \, d\Omega \tag{20}$$

where p_μ stands for the density of impulsion.

How does the entire action vary under a variation of the potential Γ?

$$\delta S = \int \delta [p_\mu v^\nu \, \Gamma^\mu_\nu + \alpha \, (\partial_\nu \Gamma^\mu_\rho - \partial_\rho \Gamma^\mu_\nu)^2 \, (1 - Tr\Gamma)] dV \, dt. \tag{21}$$

The term in α is equal to: $([\partial_\nu \Gamma^\mu_\rho - \partial_\rho \Gamma^\mu_\nu](\eta^{\nu\alpha} + 2\eta^{\nu\lambda}\Gamma^\alpha_\lambda)(\eta^{\rho\beta} + 2\eta^{\rho\kappa}\Gamma^\beta_\kappa)(\eta_{\mu\gamma} - 2\eta_{\mu\lambda}\Gamma^\lambda_\gamma)[\partial_\alpha \Gamma^\gamma_\beta - \partial_\beta \Gamma^\gamma_\alpha] \, (1 - Tr\Gamma))$.

The variation of this product of six terms can be much simplified if we consider only the first order terms in Γ in the product of $g^{\nu\alpha} = \eta^{\nu\alpha}$ and $1 - Tr\Gamma = 1$. Thus, we get:

$$\delta S = \int p_\mu v^\nu \, \delta \Gamma^\mu_\nu + \alpha \delta [(\partial_\nu \Gamma^\mu_\rho - \partial_\rho \Gamma^\mu_\nu) \, (\partial^\nu \Gamma^\rho_\mu - \partial^\rho \Gamma^\nu_\mu)] \, dVdt \tag{22}$$

or:

$$\delta S = \int p_\mu v^\nu \, \delta \Gamma^\mu_\nu + 2\alpha (\partial_\nu \Gamma^\mu_\rho - \partial_\rho \Gamma^\mu_\nu) \, \delta (\partial^\nu \Gamma^\rho_\mu - \partial^\rho \Gamma^\nu_\mu) \, dVdt \tag{23}$$

where $\partial^\nu = \eta^{\nu\mu} \partial_\mu$.

$$\delta S = \int p_\mu v^\nu \, \delta \Gamma^\mu_\nu + 2\alpha [(\partial_\nu \Gamma^\mu_\rho - \partial_\rho \Gamma^\mu_\nu) \, \partial^\nu \delta \Gamma^\rho_\mu \tag{24}$$

$$- (\partial_\nu \Gamma^\mu_\rho - \partial_\rho \Gamma^\mu_\nu) \, \partial^\rho \delta \Gamma^\nu_\mu] \, dVdt. \tag{25}$$

We swapped ∂ and δ, and by swapping ρ and ν we get:

$$\delta S = \int p_\mu v^\nu \, \delta \Gamma^\mu_\nu + 4\alpha (\partial_\rho \Gamma^\mu_\nu - \partial_\nu \Gamma^\mu_\rho) \, \partial^\rho \delta \Gamma^\nu_\mu \, dVdt. \tag{26}$$

The term in 4α is integrated by parts:

$$\int (\partial_\rho \Gamma^\mu_\nu - \partial_\nu \Gamma^\mu_\rho)\, \partial^\rho \delta\Gamma^\nu_\mu \, dVdt \tag{27}$$

$$= -\int \partial^\rho (\partial_\rho \Gamma^\mu_\nu - \partial_\nu \Gamma^\mu_\rho)\, \delta\Gamma^\nu_\mu \, dVdt$$

$$+ \int (\partial_\rho \Gamma^\mu_\nu - \partial_\nu \Gamma^\mu_\rho)\, \delta\Gamma^\nu_\mu \, dS^\rho. \tag{28}$$

The term $\int (\partial_\rho \Gamma^\mu_\nu - \partial_\nu \Gamma^\mu_\rho)\, \delta\Gamma^\nu_\mu \, dS^\rho = 0$ since $\delta\Gamma^\nu_\mu = 0$ in the time limits and $(\partial_\rho \Gamma^\mu_\nu - \partial_\nu \Gamma^\mu_\rho) = 0$ at ∞. The field strength is 0 on the boundary at ∞.

Thus we obtain:

$$\delta S = \int p_\mu v^\nu \, \delta\Gamma^\mu_\nu - 4\alpha \partial^\rho (\partial_\rho \Gamma^\mu_\nu - \partial_\nu \Gamma^\mu_\rho)\, \delta\Gamma^\nu_\mu \, dVdt. \tag{29}$$

By cancelling the variation of S and swapping μ and ν in the first term we have:

$$p_\nu v^\mu - 4\alpha \partial^\rho (\partial_\rho \Gamma^\mu_\nu - \partial_\nu \Gamma^\mu_\rho) = 0. \tag{30}$$

One corollary of eq. 30 is that the divergence of the energy impulsion tensor $(p_\nu v^\mu)$ is equal to zero. Indeed, swapping the dummy indices ν and ρ, we get:

$$\partial^\nu (p_\nu v^\mu) = 4\alpha(\partial^\nu \partial^\rho \partial_\rho \Gamma^\mu_\nu - \partial^\rho \partial^\nu \partial_\nu \Gamma^\mu_\rho)$$

$$= 4\alpha(\partial^\nu \partial^\rho \partial_\rho \Gamma^\mu_\nu - \partial^\nu \partial^\rho \partial_\rho \Gamma^\mu_\nu) = 0. \tag{31}$$

Now let us evaluate 4α. If the source current $p_\mu v^\nu$ is generated by a mass with rest density ρ $p_\mu v^\nu = \rho\, \delta^\nu_0 \delta^0_\mu c^2$ and from (30) we get:

$$\rho\, c^2 = 4\alpha[\partial_\nu \partial^\nu \Gamma^0_0 - \partial^\nu \partial_t \Gamma^0_\nu]. \tag{32}$$

With the mass density ρ at rest, the field Γ^0_ν must be static: $\partial_t \Gamma^0_\nu = 0$ and $\rho\, c^2 = 4\alpha \Delta\, \Gamma^0_0$ with Δ the Laplacian. A solution is: $\Gamma^0_0 = \frac{1}{4\pi} \int \frac{c^2 \rho}{4\alpha r} dV$ which for a mass $M = \int \rho\, dV$ and since $\Gamma^0_0 = \frac{GM}{rc^2}$, we get:

$$16\pi\alpha = \frac{c^4}{G} \left[m \frac{kg}{s^2}\right] \rightarrow 4\alpha = \frac{c^4}{4\pi G}. \tag{33}$$

Eq. 30 can be rewritten:

$$\partial^\rho (\partial_\rho \Gamma^\mu_\nu - \partial_\nu \Gamma^\mu_\rho) = \frac{4\pi G}{c^4} p_\nu v^\mu \, [m^{-2}] \tag{34}$$

where p_ν is the density of impulsion.

Depending on the values of μ, ν we have the following field equations:

1) μ, ν =0; $p_0 v^0 = \rho c^2$

$$\frac{4\pi G}{c^2} \rho = \partial_\lambda \partial^\lambda \Gamma^0_0 - \partial^\lambda \partial_t \Gamma^0_\lambda \, [m^{-2}] \tag{35}$$

and if the field is static, we get:

$$\Delta \Gamma^0_0 = \frac{4\pi G}{c^2} \rho \rightarrow \Gamma^0_0 = \frac{GM}{rc^2} \tag{36}$$

with $M = \int \rho \, dV$. The equations of motion in a central static field will be considered in the next chapter.

2) Let $\mu, \nu \neq 0$ are denoted by i, j, then for $v \ll c$, $p \simeq \rho v$ we get:

$$\frac{4\pi G}{c^4} \rho v_i v^j = \partial^\lambda (\partial_\lambda \Gamma_i^j - \partial_i \Gamma_\lambda^j) \, [m^{-2}]. \tag{37}$$

This is symmetric in i and j on the left-hand-side and can be made symmetrical in i and j on the right-hand-side because $\Gamma_i^j = \Gamma_j^i$ and by a kind of "Lorentz" condition: $\partial^\lambda \Gamma_\lambda^j = 0$.

Gravitational waves

Equation (34) can be rewritten assuming the above "Lorentz condition": $\partial^\lambda \Gamma_\lambda^j = 0$ as:

$$\partial^\rho \partial_\rho \Gamma_\nu^\mu = \frac{4\pi G}{c^4} p_\nu v^\mu. \tag{38}$$

In the vacuum:

$$\partial^\rho \partial_\rho \Gamma_\nu^\mu = 0 = \left(\frac{1}{c^2} \partial_t \partial_t - \sum_i \partial_i \partial_i\right) \Gamma_\nu^\mu$$

the field Γ_ν^μ (the massless graviton) is a wave propagating at speed of light. It can be seen as a massless particle that propagates a gravitational field.

6. The test of deflection of light by the sun

At this point we have enough results to perform the first test of this new theory, it is a kind of "stop and go" procedure as in the following scheme:

The gravitational potential of the sun is considered as central and static:

$$\Gamma_\nu^\mu = \delta_0^\mu \, \delta_\nu^0 \, \Gamma_0^0 \equiv \Gamma = \frac{GM}{rc^2}. \tag{39}$$

The impulsion of the photon is $\vec{p} = \hbar \vec{k}$ and $\vec{k} = (\omega, k_0 \infty, 0, 0) = (\omega, \omega, 0, 0)$ in the x, y plane shown

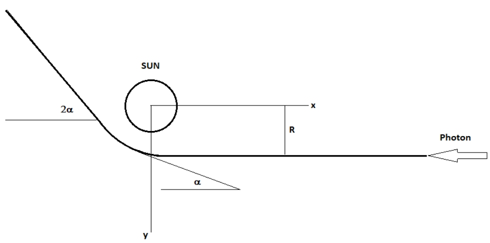

Figure 5. Deflection of light

on Fig. 5 and $\vec{v} = (1, -1, 0, 0)$ with $c = 1$; in the following calculations we also take $c = 1 = \hbar$ for clarity.

With the symbol $\dot{k} = \frac{d}{dt} k$ eq. 11 gives:

$$\dot{k_\nu} = v^\rho \left[\partial_\nu (k_\mu \Gamma_\rho^\mu) - \partial_\rho (k_\mu \Gamma_\nu^\mu)\right] \tag{40}$$

With Γ^{μ}_{ν} defined by eq. 39, it gives rise to:

$$\dot{k_\nu} = v^0\partial_\nu(k_0\Gamma) - v^\rho\partial_\rho(k_\mu\Gamma^{\mu}_{\nu}) \tag{41}$$

For $\dot{k_0}$: $\dot{k_0} = v_0\partial_t(k_0\Gamma) - v_\rho\partial_\rho(k_0\Gamma)$; $\rho = (1,2,3)$

$$\dot{k_0} = \partial_t(k_0\Gamma) + \partial_1(k_0\Gamma) = \Gamma\partial_t k_0 + k_0\partial_t\Gamma + \Gamma\partial_1 k_0 + k_0\partial_1\Gamma$$

since $\partial_t\Gamma = 0$, $\dot{k_0} = \Gamma(\partial_t k_0 + \partial_1 k_0) + k_0\partial_1\Gamma$.

With $x = x_\infty - ct = x_\infty - t$ (with c=1), $\rightarrow \partial_t = -\partial_1$ (where $\partial_1 \equiv \partial_x$). Thus,

$$\dot{k_0} = k_0\partial_1\Gamma \tag{42}$$

For $\dot{k_i}$: $\dot{k_i} = v^0\partial_i(k_0\Gamma) - v^\rho\partial_\rho(k_\mu\Gamma^{\mu}_{i}) = \partial_i(k_0\Gamma)$. Remembering that for $v_0 = 1$, $\Gamma^i_\mu = 0\,; i \neq 0$.

Along the coordinate $y = 2$, $\dot{k_2} = \partial_2(k_0\Gamma) = \partial_y(k_0\Gamma) = k_0\partial_2\Gamma$, since $\partial_2 k_0 = 0$. Thus:

$$\dot{k_2} = k_0\partial_2\Gamma \tag{43}$$

With $\dot{k_0} = k_0\partial_1\Gamma$ the time derivation of $\dot{k_2}$ reads

$$\ddot{k_2} = \partial_2(\dot{k_0}\Gamma) = (k_0\dot{\partial_2}\Gamma) = \dot{k_0}\partial_2\Gamma + k_0\partial_2\dot{\Gamma} \rightarrow \ddot{k_2} = k_0\partial_1\Gamma\partial_2\Gamma + k_0\partial_2(v^j\partial_j\Gamma).$$

The first term is in $\partial\Gamma^2$ and can be neglected versus $\partial\Gamma$ ($\Gamma \approx 10^{-6}$ at surface of the sun). Thus, $\ddot{k_2} = k_0\partial_2(v^0\partial_t\Gamma + v^1\partial_1\Gamma)$, with $v^0 = 1$, $v^1 = -1$ and $\partial_t = -\partial_1$. This leads to:

$$\ddot{k_2} = k_0\partial_2(-\partial_1\Gamma - \partial_1\Gamma) = -2k_0\partial_1\partial_2\Gamma. \tag{44}$$

Since $\Gamma = \frac{GM}{r} = \frac{GM}{\sqrt{x^2+y^2}}$ with $c = 1$

$$\partial_2\Gamma = \frac{-yGM}{(x^2+y^2)^{3/2}}$$

$$\partial_1\partial_2\Gamma = \frac{3xyGM}{(x^2+y^2)^{5/2}}$$

then

$$\ddot{k_2} = -6k_0\frac{xyGM}{(x^2+y^2)^{5/2}}. \tag{45}$$

With $\ddot{k_2} = \frac{d}{dt}\dot{k_2} = \frac{dx}{dt}\frac{d\dot{k_2}}{dx}$ and $\frac{dx}{dt} = -1$ we get $\dot{k_2} = -\int \ddot{k_2}\,dx$. Thus

$$\dot{k_2} = 6k_0GM\int\frac{xy}{(x^2+y^2)^{5/2}}\,dx. \tag{46}$$

Integrating again with $dx = -dt$ we obtain:

$$k_2 = 2k_0GMy\int_\infty^0\frac{1}{(x^2+y^2)^{3/2}}\,dx = -2k_0\frac{GMy}{y^2}. \tag{47}$$

Thus, $k_2 = -2k_0\frac{GM}{y}$ which for $y = R$ leads to

$$\frac{k_2}{k_0} = \frac{-2GM}{R} = \tan(\alpha) \simeq \alpha \tag{48}$$

still with $c = 1$. Reintroducing c we get a total deviation as $2\alpha = \frac{4GM}{Rc^2}$. This corresponds to the measured value and is the first succesful test of the validity of this new gravitational model.

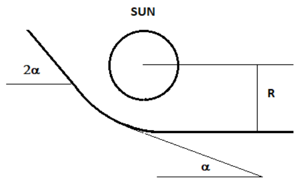

Figure 6. Total deviation

7. Equation of motion in a central static field

The basic assumptions of a central static field are: The field $\Gamma_i^j = \Gamma_0^0\,\delta_0^j\,\delta_i^0$, $\Gamma_0^0 \equiv \Gamma$ $\;\partial_t\Gamma = 0$ $\;\dot{\Gamma} = v^j\partial_j\Gamma$, and $\Gamma = \frac{GM}{rc^2}$.

Starting from eq. 11

$$\dot{p_\nu} = v^\rho[\partial_\nu(p_\mu\Gamma_\rho^\mu) - \partial_\rho(p_\mu\Gamma_\nu^\mu)].$$

For $i = 1, 2, 3$ one has:

$$\dot{p_i} = v^0\partial_i(p_0\Gamma) - v^j\partial_j(p_0\Gamma_i^0) = \partial_i(p_0\Gamma). \tag{49}$$

For i=0:

$$\dot{p_0} = v^0\partial_t(p_0\Gamma) - v^i\partial_i(p_0\Gamma). \tag{50}$$

Thus:

$$\dot{p_i} = \partial_i(p_0\Gamma) = p_0\partial_i\Gamma + \Gamma\partial_i p_0 \tag{51}$$

and

$$\dot{p_0} = -v^i\partial_i(p_0\Gamma) + \Gamma\partial_t p_0. \tag{52}$$

We now calculate $\ddot{p_i}$:

$$\ddot{p_i} = \dot{(p_0\partial_i\Gamma + \Gamma\partial_i p_0)} = \dot{p_0}\partial_i\Gamma + p_0\dot{\partial_i}\Gamma + \dot{\Gamma}\partial_i p_0 + \Gamma\partial_i\dot{p_0}. \tag{53}$$

Then replacing $\dot{p_0}$ in eq. 53 we have:

$$\ddot{p_i} = \underbrace{(-v^j\partial_j(p_0\Gamma) + \Gamma\partial_t p_0)\partial_i\Gamma}_{I} + \tag{54}$$

$$\underbrace{(p_0\partial_i(v^j\partial_j\Gamma)}_{II} + \tag{55}$$

$$\underbrace{v^j\partial_j\Gamma\partial_i p_0}_{III} + \tag{56}$$

$$\underbrace{\Gamma\partial_i(-v^j\partial_j(p_0\Gamma) + \Gamma\partial_t p_0)}_{IV}. \tag{57}$$

The term $I + IV$ gives: $\partial_i[\Gamma(-v^j\partial_j(p_0\Gamma) + \Gamma^2\partial_t p_0]$, and the term $II + III$ gives: $\partial_i[p_0 v^j \partial_j \Gamma]$, so thus,

$$\ddot{p}_i = \partial_i[p_0 v^j \partial_j \Gamma + \Gamma^2 \partial_t p_0 - \Gamma v^j \partial_j(p_0\Gamma)] \tag{58}$$

$$= \partial_i[p_0 v^j \partial_j \Gamma + \Gamma^2 \partial_t p_0 - \Gamma v^j (p_0 \partial_j \Gamma + \Gamma \partial_j p_0)] \tag{59}$$

$$= \partial_i[(p_0 v^j \partial_j \Gamma)(1 - \Gamma) + \Gamma^2(\partial_t p_0 - v^j \partial_j p_0)] \tag{60}$$

Also $\partial_t p_0 - v^j \partial_j p_0 = \dot{p}_0$ and eq. (52) gives:

$$\dot{p}_0 = \Gamma \partial_t p_0 - \Gamma v^i \partial_i p_0 - v^i p_0 \partial_i \Gamma = \Gamma \dot{p}_0 - v^i p_0 \partial_i \Gamma. \tag{61}$$

Then $\dot{p}_0(1 - \Gamma) = -p_0 v^i \partial_i \Gamma$ or $\dot{p}_0 = \frac{-p_0 v^i \partial_i \Gamma}{1-\Gamma}$.

Replacing in eq. (60) we get:

$$\ddot{p}_i = \partial_i \left[(p_0 v^j \partial_j \Gamma)(1 - \Gamma) + \frac{\Gamma^2}{1 - \Gamma}(-p_0 v^j \partial_j \Gamma) \right] \tag{62}$$

Or:

$$\ddot{p}_i = \partial_i \left[(p_0 v^j \partial_j \Gamma) \frac{1 - 2\Gamma}{1 - \Gamma} \right] ; i, j = 1, 2, 3 \tag{63}$$

Non relativistic equations of motion :

For $v \ll c$ and no external field, $p_0 \simeq mc^2$ and $p_i = \delta_{ik} v^k m = m v_i$ still for $i, j = 1, 2, 3$

Equation 63 becomes :

$$\ddot{p}_i = mc^2 v^j \partial_i \left[\frac{1 - 2\Gamma}{1 - \Gamma} \partial_j \Gamma \right] + mc^2 \partial_i v^j \left[\frac{1 - 2\Gamma}{1 - \Gamma} \partial_j \Gamma \right] \tag{64}$$

or

$$m\ddot{v}_i = mc^2 v^j \left[\frac{1 - 2\Gamma}{1 - \Gamma} \partial_i \partial_j \Gamma - \frac{\partial_i \Gamma \partial_j \Gamma}{(1 - \Gamma)^2} \right] + mc^2 \partial_i v^j \left[\frac{1 - 2\Gamma}{1 - \Gamma} \partial_j \Gamma \right] \tag{65}$$

This results in:

$$\frac{\ddot{v}_i}{c^2} = v^j \frac{1}{(1 - \Gamma)^2} [(1 - 3\Gamma + 2\Gamma^2) \partial_i \partial_j \Gamma - \partial_i \Gamma \partial_j \Gamma] + \partial_i v^j \left[\frac{1 - 2\Gamma}{1 - \Gamma} \partial_j \Gamma \right] \tag{66}$$

Equations of motion in polar coordinates r, φ

Let us now evaluate the equation of motion (66) of a body in the central static gravitational field Γ , with $\Gamma = \frac{GM}{c^2 r}$ and $\frac{\partial \Gamma}{\partial \varphi} = 0$ in the polar coordinates r, φ.

In the reference frame $\vec{e}$, $\vec{n}$, $\vec{n} = \frac{d\vec{e}}{d\varphi}$ and $\vec{e} = -\frac{d\vec{n}}{d\varphi}$. Thus, $\dot{\vec{e}} = \vec{n}\dot{\varphi}$ and $\dot{\vec{n}} = -\vec{e}\dot{\varphi}$ The velocity is given by $\vec{v} = \dot{r}\vec{e} + r\dot{\varphi}\vec{n}$. The gradient of a scalar f is $\vec{\partial} f = \frac{\partial f}{\partial r}\vec{e} + 1/r \frac{\partial f}{\partial \varphi}\vec{n}$.

Let us first evaluate the last term $\partial_i v^j \left[\frac{1-2\Gamma}{1-\Gamma} \partial_j \Gamma \right]$ in the equation 66, so that we have

$$\partial_i v^j = \partial_r(\dot{r}\vec{e_j} + r\dot{\varphi}\vec{n_j})\vec{e_i} + \frac{1}{r}\partial_\varphi(\dot{r}\vec{e_j} + r\dot{\varphi}\vec{n_j})\vec{n_i} = \dot{\varphi}\vec{n_j}\vec{e_i} \tag{67}$$

and

$$\partial_j \Gamma = \partial_r \Gamma \vec{e_j} + \frac{1}{r}\partial_\varphi \Gamma \vec{n_j} = -\frac{GM}{c^2 r^2}\vec{e_j}. \tag{68}$$

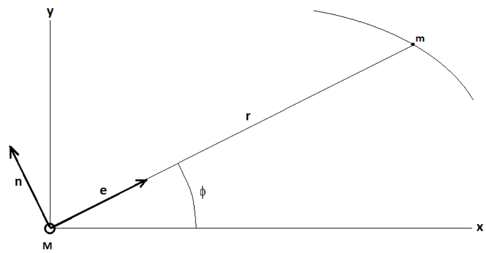

Figure 7. polar coordinates

This term contains the factor $\vec{e_j}\,\vec{n_j}\,\vec{e_i} = 0$ since $(\vec{e}|\vec{n}) = 0$. Thus, when developing eq. 66 we are left with two terms: Term I = $\underbrace{(1-3\Gamma+2\Gamma^2)v^j\partial_j\partial_i\Gamma)}_{I}$ and term II $=\underbrace{(-v^j\partial_j\Gamma\partial_i\Gamma)}_{II}$.

Developing them we have (we leave the arrow on top of e and n for easier identification)

Term I: The calculation of $v^j\partial_j\partial_i\Gamma$ gives rise to

$$v^j\partial_j\partial_i\Gamma = (\dot r\vec{e_j} + r\dot\varphi\vec{n_j}\,|\,\frac{2GM}{c^2r^3}\vec{e_j}\,\vec{e_i} - \frac{GM}{c^2r^3}\vec{n_j}\,\vec{n_i}). \tag{69}$$

Indeed:

$$\partial_j\partial_i\Gamma = \frac{\partial(\partial_i\Gamma)}{\partial r}\vec{e_j} + 1/r\frac{\partial(\partial_i\Gamma)}{\partial\varphi}\vec{n_j} = \partial_r(\frac{-GM}{c^2r^2}\vec{e_i})\vec{e_j} + 1/r\,\partial_\varphi(\frac{-GM}{c^2r^2}\vec{e_i})\vec{n_j} \tag{70}$$

$$= \frac{2GM}{c^2r^3}\vec{e_i}\,\vec{e_j} - \frac{GM}{c^2r^2}(\partial_r\vec{e_i})\vec{e_j} + 1/r\left(\partial_\varphi(\frac{-GM}{c^2r^2})\vec{e_i}\,\vec{n_j} - \frac{GM}{c^2r^2}(\partial_\varphi\vec{e_i})\vec{n_j}\right)$$

$$= \frac{2GM}{c^2r^3}\vec{e_i}\,\vec{e_j} - \frac{GM}{c^2r^3}\vec{n_i}\,\vec{n_j} \tag{71}$$

since $\partial_i\Gamma = \partial_r\Gamma\vec{e_i} + \frac{1}{r}\partial_\varphi\Gamma\vec{n_i}$, $\partial_r\Gamma = \frac{-GM}{c^2r^2}$ and $\partial_\varphi\Gamma = 0$.

Then eq. 69 becomes:

$$v^j\partial_j\partial_i\Gamma = \dot r\frac{2GM}{c^2r^3}(\vec{e_j}|\vec{e_j})\vec{e_i} - \dot r\frac{GM}{c^2r^3}(\vec{e_j}|\vec{n_j})\vec{n_i} + r\dot\varphi\frac{2GM}{c^2r^3}(\vec{n_j}|\vec{e_j})\vec{e_i} - r\dot\varphi\frac{GM}{c^2r^3}(\vec{n_j}|\vec{n_j})\vec{n_i} \tag{72}$$

Since $(\vec{e}|\vec{e}) = (\vec{n}|\vec{n}) = 1$ and $(\vec{e}|\vec{n}) = 0$ we get:

$$v^j\partial_j\partial_i\Gamma = \frac{2\dot rGM}{c^2r^3}\vec{e_i} - \dot\varphi\frac{GM}{c^2r^2}\vec{n_i}$$

And term I becomes (with $\Gamma = \frac{GM}{c^2r}$)

$$(1-3\Gamma+2\Gamma^2)\,v^j\partial_j\partial_i\Gamma$$

$$= (2\dot r\frac{GM}{c^2r^3} - 3a\frac{GM}{c^2r}\frac{2\dot rGM}{c^2r^3} + 2\frac{G^2M^2}{c^4r^2}\frac{2\dot rGM}{c^2r^3})\vec{e_i}$$

$$+ (-\dot\varphi\frac{GM}{c^2r^2} + 3\frac{GM}{c^2r}\dot\varphi\frac{GM}{c^2r^2} - 2\frac{G^2M^2}{c^4r^2}\dot\varphi\frac{GM}{c^2r^2})\vec{n_i}. \tag{73}$$

Term II:

$$(-v^j\partial_j\Gamma\partial_i\Gamma) = -(\dot{r}\vec{e_j} + r\dot{\varphi}\vec{n_j} \mid \frac{-GM}{c^2r^2}\vec{e_j})\frac{-GM}{c^2r^2}\vec{e_i} = -\dot{r}\frac{G^2M^2}{c^4r^4}\vec{e_i}. \tag{74}$$

Collecting both terms *I* + *II* we get:

$$\frac{\ddot{\vec{v}}}{c^2} = \frac{1}{(1-\frac{GM}{c^2r})^2}\left[\left(2\dot{r}\frac{GM}{c^2r^3} - 7\dot{r}\frac{G^2M^2}{c^4r^4} + 4\dot{r}\frac{G^3M^3}{c^6r^5}\right)\vec{e} + \left(-\dot{\varphi}\frac{GM}{c^2r^2} + 3\dot{\varphi}\frac{G^2M^2}{c^4r^3} - 2\dot{\varphi}\frac{G^3M^3}{c^6r^4}\right)\vec{n}\right] \tag{75}$$

$$\frac{\ddot{\vec{v}}}{c^2} = \frac{1}{(1-\frac{GM}{c^2r})^2}\left[\left(-\frac{G\dot{M}}{c^2r^2} + 7/3\frac{G^2\dot{M}^2}{c^4r^3} - \frac{G^3\dot{M}^3}{c^6r^4}\right)\vec{e}\right. \tag{76}$$

$$\left. + \left(-\frac{GM}{c^2r^2} + 3\frac{G^2M^2}{c^4r^3} - 2\frac{G^3M^3}{c^6r^4}\right)\dot{\varphi}\vec{n}\right]. \tag{77}$$

Since $\dot{\varphi}\vec{n} = \dot{\vec{e}}$ we get (we postpone terms in $\vec{e}$ for the subsequent steps)

$$\frac{\ddot{\vec{v}}}{c^2} = \frac{1}{\left(1-\frac{GM}{c^2r}\right)^2}\left[\left(\frac{-G\dot{M}}{c^2r^2} + \frac{9}{3}\frac{G^2\dot{M}^2}{c^4r^3} - \frac{2}{3}\frac{G^2\dot{M}^2}{c^4r^3} - \frac{2G^3\dot{M}^3}{c^6r^4} + \frac{G^3\dot{M}^3}{c^6r^4}\right)\vec{e}\right. \tag{78}$$

$$\left. + \left(\frac{-GM}{c^2r^2} + 3\frac{G^2M^2}{c^4r^3} - 2\frac{G^3M^3}{c^6r^4}\right)\dot{\vec{e}}\right]. \tag{79}$$

$$\frac{\ddot{\vec{v}}}{c^2} = \frac{1}{\left(1-\frac{GM}{c^2r}\right)^2}\frac{d}{dt}\left[\left(\frac{-GM}{c^2r^2} + 3\frac{G^2M^2}{c^4r^3} - 2\frac{G^3M^3}{c^6r^4}\right)\vec{e}\right] \tag{80}$$

$$+\frac{1}{\left(1-\frac{GM}{c^2r}\right)^2}(-2/3\frac{G^2\dot{M}^2}{c^4r^3} + \frac{G^3\dot{M}^3}{c^6r^4})\vec{e}. \tag{81}$$

The last term in 81 can be shown to be equal to:

$$\left(-\frac{GM}{c^2r^2} + 3\frac{G^2M^2}{c^4r^3} - 2\frac{G^3M^3}{c^6r^4}\right)\vec{e}\frac{d}{dt}\frac{1}{\left(1-\frac{GM}{c^2r}\right)^2}$$

Thus we finally get:

$$\frac{\ddot{\vec{v}}}{c^2} = \frac{d}{dt}\left[\frac{1}{\left(1-\frac{GM}{c^2r}\right)^2}\left(\frac{-GM}{c^2r^2} + 3\frac{G^2M^2}{c^4r^3} - 2\frac{G^3M^3}{c^6r^4}\right)\vec{e}\right]. \tag{82}$$

Integrating on time we get the gravitational acceleration (the integration constant can be set to 0 in a suitable reference frame)

$$\frac{\dot{\vec{v}}}{c^2} = \frac{1}{\left(1-\frac{GM}{c^2r}\right)^2}\left(-\frac{GM}{c^2r^2} + 3\frac{G^2M^2}{c^4r^3} - 2\frac{G^3M^3}{c^6r^4}\right)\vec{e}. \tag{83}$$

To the first order in G we find **Newton's law**:

$$\vec{F} = m\dot{\vec{v}} = \frac{-GmM}{r^2}\vec{e} \tag{84}$$

and also the **Equivalence principle** stating that the effects of a gravitational field are identical to an acceleration given by eq. (83) .

It is interesting to note that the second term in (83) is a **repulsive acceleration** $3\frac{G^2M^2}{c^4r^3}$. It is usually very small compared to the first term. For instance at the surface of earth, with: $M = 6\,10^{24}\,kg, r = 6,37\,10^6 m, G = 6,67\,10^{-11}\,m^3/kgs^2, c = 3\,10^8 m/s$, the ratio of the repulsive force to the main attractive force is $\frac{3GM}{c^2r} \simeq 2\,10^{-9}$. Could this repulsive force be measured ? Possibly by comparing the velocities of satellites at different altitudes. However, it remains open right now.

***Solution in the polar coordinates** r, φ*

The radial $\vec{e}$ component of the acceleration $\dot{\vec{v}}$ is:

$$\frac{d^2r}{dt^2} - r(\frac{d\varphi}{dt})^2$$

And its tangential $\vec{n}$ component is:

$$r\frac{d^2\varphi}{dt^2} + 2\frac{dr}{dt}\frac{d\varphi}{dt} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\varphi}).$$

We thus get the following set of equations:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 = c^2\left[\left(1 - \frac{GM}{c^2r}\right)^{-2}\left(-\frac{GM}{c^2r^2} + 3\frac{G^2M^2}{c^4r^3} - 2\frac{G^3M^3}{c^6r^4}\right)\right] \tag{85}$$

and

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\varphi}) = 0. \tag{86}$$

Eq. (86) expresses the conservation of angular momentum and gives $r^2\dot{\varphi} = h$ with h a constant (m^2/s).

Regarding eq. (85), we must first develop $\frac{d^2r}{dt^2}$ as follows

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} \rightarrow \dot{r} = \frac{h}{r^2}\frac{dr}{d\varphi}$$

$$\ddot{r} = \frac{d\dot{r}}{d\varphi}\frac{d\varphi}{dt} = \frac{h^2}{r^4}\frac{d^2r}{d\varphi^2} - 2\frac{h^2}{r^5}(\frac{dr}{d\varphi})^2. \tag{87}$$

Eq (85) becomes

$$\frac{h^2}{r^4}\frac{d^2r}{d\varphi^2} - 2\frac{h^2}{r^5}(\frac{dr}{d\varphi})^2 - \frac{h^2}{r^3} - \left(1 - \frac{GM}{c^2r}\right)^{-2}\left(-\frac{GM}{r^2} + 3\frac{G^2M^2}{c^2r^3} - 2\frac{G^3M^3}{c^4r^4}\right) = 0. \tag{88}$$

And after multiplying by r^2:

$$\frac{h^2}{r^2}\frac{d^2r}{d\varphi^2} - 2\frac{h^2}{r^3}\left(\frac{dr}{d\varphi}\right)^2 - \frac{h^2}{r} - \left(1 - \frac{GM}{c^2r}\right)^{-2}\left(-GM + 3\frac{G^2M^2}{c^2r} - 2\frac{G^3M^3}{c^4r^2}\right) = 0. \tag{89}$$

Let $u = \frac{1}{r}$ then $\frac{du}{d\varphi} = -\frac{1}{r^2}\frac{dr}{d\varphi}$ and $\frac{d^2u}{d\varphi^2} = -\frac{1}{r^2}\frac{d^2r}{d\varphi^2} + \frac{2}{r^3}(\frac{dr}{d\varphi})^2$. So that we get:

$$-h^2\frac{d^2u}{d\varphi^2} - h^2u + \left(1 - \frac{GMu}{c^2}\right)^{-2}\left(GM - \frac{3G^2M^2u}{c^2} + \frac{2G^3M^3u^2}{c^4}\right) = 0. \tag{90}$$

Or

$$-h^2\frac{d^2u}{d\varphi^2} - h^2u + GM\left(\frac{1 - \frac{2GMu}{c^2} + \frac{G^2M^2u^2}{c^4} - \frac{GMu}{c^2} + \frac{G^2M^2u^2}{c^4}}{1 - \frac{2GMu}{c^2} + \frac{G^2M^2u^2}{c^4}}\right) = 0$$

Dividing by h^2 and writing $\frac{du}{d\varphi} \equiv u'$ and rearranging, we get

$$u'' + u - \frac{GM}{h^2}\left(1 - \frac{\frac{GMu}{c^2}}{1 - \frac{GMu}{c^2}}\right) = 0$$

Or

$$u'' + u\left(1 + \frac{G^2M^2}{c^2h^2}\left(1 - \frac{GMu}{c^2}\right)^{-1}\right) = \frac{GM}{h^2}. \tag{91}$$

Eq. (91) is the equation of the orbit of a body around a stationary body of a mass M for non-relativistic speed.

Relativistic equations of motion of a body in a central static field

We rewrite eq. (63) where we will use the relativistic impulsion $\vec{p}$ and p_0:

$$\ddot{p}_i = \partial_i\left[p_0v^j\frac{1 - 2\Gamma}{1 - \Gamma}\partial_j\Gamma\right]. \tag{92}$$

Which with the relativistic impulsion of a body according to Landau and Lifschitz 1964, gives

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{93}$$

and

$$p_0 = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{94}$$

Then, one gets:

$$\ddot{p}_i = p_0v^j\partial_i\left[\frac{1 - 2\Gamma}{1 - \Gamma}\partial_j\Gamma\right] + \left(\frac{1 - 2\Gamma}{1 - \Gamma}\partial_j\Gamma\right)v^j\partial_ip_0 \tag{95}$$

The first term of eq. (95) is:

$$\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} v^j \partial_i \left[\frac{1-2\Gamma}{1-\Gamma}\partial_j\Gamma\right] = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} v^j \frac{1}{(1-\Gamma)^2}\left[(1-3\Gamma+2\Gamma^2)\partial_i\partial_j\Gamma - \partial_i\Gamma\partial_j\Gamma\right] \tag{96}$$

$$= \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d}{dt}\left[\frac{1}{\left(1-\frac{GM}{c^2r}\right)^2}\left(\frac{-GM}{c^2r^2}+3\frac{G^2M^2}{c^4r^3}-2\frac{G^3M^3}{c^6r^4}\right)\vec{e}\right] \tag{97}$$

which is obtained in the same way as eq. (82) was obtained from eq. (66).

The second term of eq. (95) is

$$(\frac{1-2\Gamma}{1-\Gamma}\partial_j\Gamma)v^j\partial_i p_0 = (\frac{1-2\Gamma}{1-\Gamma})\partial_j\Gamma\, v^j mc^2\,\partial_i\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Which in polar coordinates reads

$$\partial_j\Gamma\, v^j = (\dot{r}\vec{e_j} + r\dot{\varphi}\vec{n_j}|\partial_r(\frac{GM}{c^2r})\vec{e_j} + \frac{1}{r}\partial_\varphi\Gamma\vec{n_j}) = -\dot{r}\frac{GM}{c^2r^2}$$

since $\partial_\varphi\Gamma = 0$ and $\partial_i\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{2c^2}(1-\frac{v^2}{c^2})^{-3/2}\partial_i(-v^2)$.

But $\partial_i(v^2) = \partial_r(\dot{r}^2 + r^2\dot{\varphi}^2)\vec{e_i} + \frac{1}{r}\partial_\varphi(\dot{r}^2 + r^2\dot{\varphi}^2)\vec{n_i}$ and $= 2r\dot{\varphi}^2\vec{e_i}$ since $\partial_r(\dot{r}^2) = 0 = \partial_\varphi(\dot{\varphi}^2)$ which after combining gives the second term as

$$-mc^2(1-\frac{v^2}{c^2})^{-3/2}\frac{1-2\Gamma}{1-\Gamma}\frac{1}{c^2}r\dot{r}\dot{\varphi}^2\frac{GM}{c^2r^2}\vec{e_i}$$

$$= -mc^2(1-\frac{v^2}{c^2})^{-3/2}\frac{1-3\Gamma+2\Gamma^2}{(1-\Gamma)^2}\frac{1}{c^2}\dot{r}\dot{\varphi}^2\Gamma\vec{e_i}$$

$$= -\frac{1}{(1-\frac{GM}{c^2r})^2}(\frac{GM}{c^2r} - 3\frac{G^2M^2}{c^4r^2} + 2\frac{G^3M^3}{c^6r^3})\frac{1}{c^2}mc^2(1-\frac{v^2}{c^2})^{-3/2}\dot{r}\dot{\varphi}^2\vec{e_i}$$

$$= \frac{1}{(1-\frac{GM}{c^2r})^2}(\frac{-GM}{c^2r^2} + 3\frac{G^2M^2}{c^4r^3} - 2\frac{G^3M^3}{c^6r^4})\frac{1}{c^2}mc^2(1-\frac{v^2}{c^2})^{-3/2}r\dot{r}\dot{\varphi}^2\vec{e_i}.$$

and since $\partial_i(v^2) = 2r\dot{\varphi}^2\vec{e_i}$, we have: $\dot{r}2r\dot{\varphi}^2\vec{e_i} = \dot{r}\partial_i(v^2)\vec{e_i} = \dot{r}\partial_r(v^2)\vec{e_i} = \frac{dv^2}{dt}\vec{e_i}$ from which it follows that $\frac{1}{c^2}mc^2(1-\frac{v^2}{c^2})^{-3/2}r\dot{r}\dot{\varphi}^2\vec{e_i} = \frac{d}{dt}\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\vec{e_i}$. Combining the two terms, we get

$$\ddot{\vec{p}} = \frac{d}{dt}\left[\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1}{\left(1-\frac{GM}{c^2r}\right)^2}\left(\frac{-GM}{c^2r^2}+3\frac{G^2M^2}{c^4r^3}-2\frac{G^3M^3}{c^6r^4}\right)\vec{e}\right].$$

Then integrating on time gives

$$\dot{\vec{p}} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1}{\left(1-\frac{GM}{c^2r}\right)^2}\left(\frac{-GM}{c^2r^2}+3\frac{G^2M^2}{c^4r^3}-2\frac{G^3M^3}{c^6r^4}\right)\vec{e}. \tag{98}$$

This can be compared with eq. (83). Let us look at the left hand term of that equation where there appears the time derivative of the relativistic impulsion. From the definition (93) we evaluate the time derivative of the relativistic impulsion as follows.

$$\frac{d}{dt}\vec{p} = m\left(\frac{\dot{\vec{v}}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{(\vec{v}|\dot{\vec{v}})\,\vec{v}}{c^2(1-\frac{v^2}{c^2})^{3/2}}\right). \tag{99}$$

Note that according to Landau and Lifschitz 1964, when the force is normal to speed we would have

$$\frac{d}{dt}\vec{p} = m\left(\frac{\dot{\vec{v}}}{\sqrt{1-\frac{v^2}{c^2}}}\right) \tag{100}$$

and when the force and speed are co-linear we would get

$$\frac{d}{dt}\vec{p} = m\left(\frac{\dot{\vec{v}}}{(1-\frac{v^2}{c^2})^{3/2}}\right). \tag{101}$$

In the following, since forces and speeds can have independent orientations we have to use the general formula as in eq. (99) for the time derivative of impulsion. Consequently in polar coordinates we obtain

$$\dot{\vec{v}} = (\ddot{r} - r\dot{\varphi}^2)\,\vec{e} + \frac{1}{r}(\dot{r^2\dot{\varphi}})\,\vec{n}$$

and

$$(\vec{v}|\dot{\vec{v}}) = \frac{1}{2}\dot{v^2} = \frac{1}{2}(\dot{r}^2 + \dot{r^2\varphi^2}).$$

Then eqs. 98 and 99 result in the following cases.

Along $\vec{n}$

$$\frac{\frac{1}{r}(\dot{r^2\dot{\varphi}})}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{r\dot{\varphi}(\dot{v^2})}{2c^2(1-\frac{v^2}{c^2})^{3/2}} = 0 \tag{102}$$

and along $\vec{e}$ (dividing both by m)

$$\frac{\ddot{r} - r\dot{\varphi}^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{\dot{r}\,\dot{v^2}}{2c^2(1-\frac{v^2}{c^2})^{3/2}} = \frac{c^2}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1}{\left(1-\frac{GM}{c^2r}\right)^2}\left(\frac{-GM}{c^2r^2} + 3\frac{G^2M^2}{c^4r^3} - 2\frac{G^3M^3}{c^6r^4}\right). \tag{103}$$

Eq. (102) represents the relativistic conservation of angular momentum, after the following development. Multiply both by $\sqrt{1-\frac{v^2}{c^2}}$ so that $\Rightarrow (\dot{r^2\dot{\varphi}}) = -r^2\ \dot{\varphi}\ \frac{\dot{v^2}}{2(c^2-v^2)}$. Thus,

$$\frac{(\dot{r^2\dot{\varphi}})}{r^2\dot{\varphi}} = -\frac{\dot{v^2}}{2(c^2-v^2)} = \frac{1}{2}\frac{(\dot{c^2 - v^2})}{(c^2-v^2)}. \tag{104}$$

Integrating on time gives (the units are again more physical in this calculation)

$$\ln(r^2\dot{\varphi}) = \frac{1}{2}\ln(c^2 - v^2) + \kappa \tag{105}$$

with κ a constant, and after exponentiation and defining $e^{\kappa} \equiv h/c$ we obtain

$$r^2\dot{\varphi} = \sqrt{(c^2 - v^2)}\, e^{\kappa} = \sqrt{(c^2 - v^2)}\, h/c = h\sqrt{1 - \frac{v^2}{c^2}}. \tag{106}$$

For $v \ll c$ one gets the classical form $r^2\dot{\varphi} = h$ however for the calculation of the precession of the perihelion of Mercury we will use the relativistic equation eq. 106.

Let us now develop eq. 103 such that it will give us the relativistic relation for the orbit of a body around a stationary mass M. After multiplication of both terms of (103) by $\sqrt{1 - \frac{v^2}{c^2}}$ we get:

$$\ddot{r} - r\dot{\varphi}^2 + \frac{\dot{r}(\dot{v^2})}{2(c^2 - v^2)} = c^2 \frac{1}{\left(1 - \frac{GM}{c^2 r}\right)^2} \left(\frac{-GM}{c^2 r^2} + 3\frac{G^2M^2}{c^4 r^3} - 2\frac{G^3M^3}{c^6 r^4}\right). \tag{107}$$

We now expand the first term of (107) so that r depends on φ, using the relations $\dot{\varphi} = \frac{h}{r^2}\sqrt{1 - \frac{v^2}{c^2}}$ and $\dot{r} = \frac{dr}{d\varphi}\dot{\varphi} = r'\frac{h}{r^2}\sqrt{1 - \frac{v^2}{c^2}}$ Then, let us calculate $\ddot{r}$.

$$\ddot{r} = \frac{d\dot{r}}{d\varphi}\dot{\varphi} = \frac{h}{r^2}\sqrt{1 - \frac{v^2}{c^2}} \left(\frac{d}{d\varphi}(\frac{h}{r^2}\sqrt{1 - \frac{v^2}{c^2}}\frac{dr}{d\varphi})\right) =$$

$$\frac{h^2}{r^2}\sqrt{1 - \frac{v^2}{c^2}} \left(\frac{-2}{r^3} r'^2 \sqrt{1 - \frac{v^2}{c^2}} + r'\frac{1}{r^2}\frac{d}{d\varphi}\sqrt{1 - \frac{v^2}{c^2}} + \frac{\sqrt{1 - \frac{v^2}{c^2}}}{r^2} r''\right)$$

$$\ddot{r} = \frac{h^2}{r^2}\left(\frac{-2}{r^3} r'^2 (1 - \frac{v^2}{c^2}) + \frac{r'}{r^2}\sqrt{1 - \frac{v^2}{c^2}}\frac{d}{d\varphi}\sqrt{1 - \frac{v^2}{c^2}} + \frac{1 - \frac{v^2}{c^2}}{r^2} r''\right) \tag{108}$$

Now $\sqrt{1 - \frac{v^2}{c^2}}\frac{d}{d\varphi}\sqrt{1 - \frac{v^2}{c^2}} = 1/2\frac{d}{d\varphi}(1 - \frac{v^2}{c^2}) = 1/2\frac{d}{d\varphi}(-\frac{v^2}{c^2})$ and $\frac{d}{d\varphi}v^2 = \dot{v^2}\frac{dt}{d\varphi} = \dot{v^2}\frac{r^2}{h\sqrt{1 - \frac{v^2}{c^2}}}$.

After replacing by the above in (108) we get:

$$\ddot{r} = (1 - \frac{v^2}{c^2})(\frac{-2h^2 r'^2}{r^5}) - \frac{hr'\dot{v^2}}{2r^2c^2\sqrt{1 - \frac{v^2}{c^2}}} + (1 - \frac{v^2}{c^2})\frac{h^2 r''}{r^4}. \tag{109}$$

The two other terms in (107) are

$$-r\dot{\varphi^2} = \frac{-h^2}{r^3}(1 - \frac{v^2}{c^2}) \tag{110}$$

and

$$\frac{\dot{r}\dot{v^2}}{2c^2(1 - \frac{v^2}{c^2})} = \frac{hr'\dot{v^2}}{2r^2c^2\sqrt{1 - \frac{v^2}{c^2}}} \tag{111}$$

using $\dot{r} = \frac{dr}{d\varphi}\frac{d\varphi}{dt}$ and $\dot{\varphi} = \frac{h}{r^2\sqrt{1 - \frac{v^2}{c^2}}}$.

So the first term of (107) $\ddot{r} - r\dot{\varphi}^2 + \frac{\dot{r}(\dot{v^2})}{2(c^2-v^2)}$ becomes:

$$(1 - \frac{v^2}{c^2})(\frac{-2h^2r'^2}{r^5} + \frac{h^2r''}{r^4} - \frac{h^2}{r^3}). \tag{112}$$

Dividing both terms of (107) by $(1 - \frac{v^2}{c^2})$ we get

$$\frac{-2h^2r'^2}{r^5} + \frac{h^2r''}{r^4} - \frac{h^2}{r^3} = \frac{1}{(1-\frac{v^2}{c^2})\left(1 - \frac{GM}{c^2r}\right)^2}\left(\frac{-GM}{r^2} + 3\frac{G^2M^2}{c^2r^3} - 2\frac{G^3M^3}{c^4r^4}\right). \tag{113}$$

We now apply the same analysis to (113) as we did for eq. 89. After multiplying in (113) by r^2 we have

$$\frac{-2h^2r'^2}{r^3} + \frac{h^2r''}{r^2} - \frac{h^2}{r} - \frac{1}{(1-\frac{v^2}{c^2})\left(1 - \frac{GM}{c^2r}\right)^2}\left(-GM + 3\frac{G^2M^2}{c^2r} - 2\frac{G^3M^3}{c^4r^2}\right) = 0. \tag{114}$$

Let $u = \frac{1}{r}$ then $\frac{du}{d\varphi} = -\frac{1}{r^2}\frac{dr}{d\varphi}$ and $\frac{d^2u}{d\varphi^2} = -\frac{1}{r^2}\frac{d^2r}{d\varphi^2} + \frac{2}{r^3}(\frac{dr}{d\varphi})^2$. So, we get the following differential equation in u

$$-h^2\frac{d^2u}{d\varphi^2} - h^2u + \left(1 - \frac{GMu}{c^2}\right)^{-2}(1 - \frac{v^2}{c^2})^{-1}\left(GM - \frac{3G^2M^2u}{c^2} + \frac{2G^3M^3u^2}{c^4}\right) = 0 \tag{115}$$

or

$$-h^2\frac{d^2u}{d\varphi^2} - h^2u + (1 - \frac{v^2}{c^2})^{-1}GM\left(\frac{1 - \frac{2GMu}{c^2} + \frac{G^2M^2u^2}{c^4} - \frac{GMu}{c^2} + \frac{G^2M^2u^2}{c^4}}{1 - \frac{2GMu}{c^2} + \frac{G^2M^2u^2}{c^4}}\right) = 0.$$

Dividing by h^2, writing $\frac{du}{d\varphi} \equiv u'$ and rearranging we get

$$u'' + u - \frac{GM}{h^2(1-\frac{v^2}{c^2})}\left(1 - \frac{\frac{GMu}{c^2}}{1 - \frac{GMu}{c^2}}\right) = 0 \tag{116}$$

or else

$$u'' + u\left(1 + \frac{G^2M^2}{c^2h^2(1-\frac{v^2}{c^2})(1-\frac{GMu}{c^2})}\right) = \frac{GM}{h^2(1-\frac{v^2}{c^2})}. \tag{117}$$

This is the polar equation of the orbit of a body (a planet) around a stationary body of mass M for relativistic speed. Compared with eq. (91), it contains a relativistic factor $(1 - \frac{v^2}{c^2})$ which has some dependance on u. We try to solve this differential equation.

Let us find a relation between $(1 - \frac{v^2}{c^2})$ and u. Observe that both $1/(1 - \frac{v^2}{c^2})$ and u are periodic in φ on a classical non relativistic orbit, see Figs. 8 and 10. We define the relation between $1/(1 - \frac{v^2}{c^2})$ and u as follows.

$$\frac{1}{(1-\frac{v^2}{c^2})} = A\frac{u}{(\frac{GM}{h^2})} + B \tag{118}$$

where u is dimensionless via the factor $1/\frac{GM}{h^2}$. A and B can be derived from the values of $1/(1 - \frac{v^2}{c^2})$ and u at the perihelion and aphelion of the planet, the result is, with e being the eccentricity A= $\frac{2G^2M^2}{c^2h^2}$ and B= $1 - \frac{G^2M^2(1-e^2)}{c^2h^2}$.

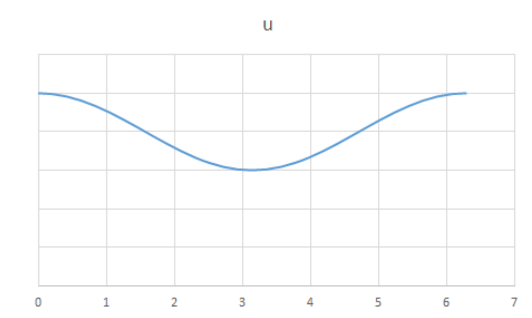

Figure 8. $u(\varphi)$

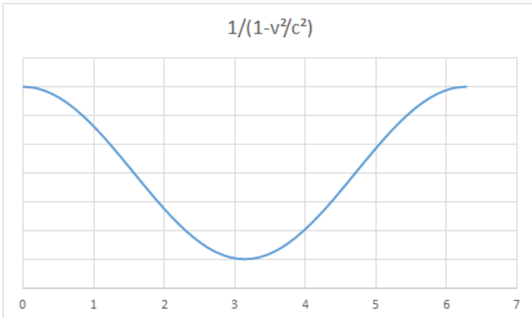

Figure 9. $1/(1-\frac{v^2}{c^2})(\varphi)$

$1/(1-\frac{v^2}{c^2})$ can be written as

$$\frac{1}{(1-\frac{v^2}{c^2})} = (\frac{2GMu}{c^2} + 1 - \frac{G^2M^2(1-e^2)}{c^2h^2}) \tag{119}$$

For non relativistic speed $v << c$ it holds $\frac{1}{(1-\frac{v^2}{c^2})} \approx (1+\frac{v^2}{c^2})$ and we get the conservation of energy: $\frac{mv^2}{2} = \frac{mGM}{r} - \frac{G^2M^2(1-e^2)m}{2h^2}$, i.e., the sum of kinetic and potentail energy is a constant.

Then, the eq. 117 becomes

$$u'' + u\left(1 + \frac{G^2M^2}{c^2h^2(1-\frac{GMu}{c^2})}(\frac{2GMu}{c^2} + 1 - \frac{G^2M^2(1-e^2)}{c^2h^2})\right) = \frac{2G^2M^2u}{c^2h^2} + \frac{GM}{h^2} - \frac{G^3M^3(1-e^2)}{c^2h^4} \tag{120}$$

8. The Schwarzschild metric

In order to calculate the precession of the perihelion of Mercury, we need to express eq. 120 in the Schwarzschild metric of the proper coordinates of Mercury and not in the Minkowskian metric used up to now that corresponds to an observer situated at infinite distance from the massive body and where Γ tends to zero.

In the spherical coordinates a gravitational field can be written to the first order as $\Gamma = \frac{GM}{rc^2}$ (36) and the Schwartszchild metric is expressed as $d\tau^2 = (1-2\Gamma)dt^2 - (1-2\Gamma)^{-1}dr^2 - r^2d\theta^2 - r^2\sin\theta^2 d\phi^2$.

Which gives for $\Gamma << 1$ in a Minkowskian metric: $d\tau^2 = dt'^2 - dr'^2 - r^2 d\theta^2 - r^2 \sin\theta^2 d\phi^2$ with the change of variables $dt' = (1-\Gamma)dt$, $dr' = (1+\Gamma)dr$.

Then $\ddot{r}$ in eq. 107 becomes $\frac{1-\Gamma}{1+2\Gamma}\frac{d^2r'}{dt'^2} = (1-3\Gamma)\frac{d^2r'}{dt'^2}$. The $(1-3\Gamma)$ factor applies to the first term of (107) and eq. 115 becomes

$$(1-3\Gamma)(u''+u) - (1+4\Gamma)\frac{GM}{h^2}(1-3\Gamma+2\Gamma^2) = 0. \tag{121}$$

Recalling that $u = 1/r$, u'' is the second derivative of u by ϕ and $\Gamma = GMu/c^2$. We also need to consider the modification of the Laplacian in (36) that generates Γ; Δ becomes $(1+2\Gamma)\Delta$ and the term $GM(1-3\Gamma+2\Gamma^2)$ must be multiplied by $(1+2\Gamma)\Delta$. Thus, the final form of eq. 115 is

$$(1-3\Gamma)(u''+u) - (1+4\Gamma)\frac{GM}{h^2}(1+2\Gamma)(1-3\Gamma+2\Gamma^2) = 0 \tag{122}$$

which can be simplified by eliminating the higher order terms

$$(u''+u) - (1+6\Gamma)\frac{GM}{h^2} = 0. \tag{123}$$

Precession of the perihelion of Mercury

We now look for a solution of eq. 123 in the form $u = \mu + \alpha\cos(\beta\varphi)$ with $\alpha = \frac{eGM}{h^2} = \frac{(u_1-u_2)}{2}$ and $\mu = \frac{(u_1+u_2)}{2}$. Eq. 123 can be written

$$u'' + u = (1 + 6\,GMu/c^2)\frac{GM}{h^2} \tag{124}$$

or else

$$u'' + u(1 - 6\,G^2M^2/h^2c^2) = \frac{GM}{h^2}. \tag{125}$$

Replacing u and solving for the term $\cos(\beta\varphi)$ we get

$$-\beta^2\alpha\cos(\beta\varphi) + (1 - 6\frac{G^2M^2}{c^2h^2})\alpha\cos(\beta\varphi) = 0. \tag{126}$$

Thus $\beta^2 = \left[1 - 6\frac{G^2M^2}{c^2h^2}\right]$ and $\beta \simeq 1 - 3\frac{G^2M^2}{c^2h^2}$ we get for the homogeneous solution

$$u = \alpha\cos\left[\left(1 - 3\frac{G^2M^2}{c^2h^2}\right)\varphi\right]. \tag{127}$$

What is the resulting advance of the perihelion? With $M = 2\,10^{30}\,kg$ the mass of the sun, $G = 6.67\,10^{-11} m^3/kgs^2$, $r = 1/u = 57.9\,10^9 m$ is the average distance of Mercury to the Sun, $e = 0.204$ for Mercury, $h = 2.7\,10^{15} m^2/s$ for Mercury, $c = 3\,10^8 m/s$. The period of revolution of Mercury = 88 days. We get a phase shift due to the $3\frac{G^2M^2}{c^2h^2}$ term. This shift is $\frac{6\pi G^2M^2}{c^2h^2} = 5.11\,10^{-7}$ radians per revolution. This corresponds to 43.2″ per century. The corresponding change in the position of the perihelion moves forward to the orbit of Mercury, the accepted value up to now is indeed 43″ in the same direction.

9. Black holes

Objects whose gravitational fields are too strong for light to escape were already considered in the 18th century by John Michell and Pierre-Simon Laplace. When described by general relativity, the black hole contains a gravitational singularity at the origin, a region where the spacetime curvature becomes infinite and contains all the mass of the black hole.

First we will limit ourselves to the study of black holes having mass M with no electric charge and no angular momentum. What happens to a body in the vicinity r of a black hole? Will it be swallowed and disappear forever?

Let r be the distance of this object to the centre of the black hole, which is supposed to be located at r=0 and to contain the mass M. We don't know yet if the large body of mass M is a black hole or not, at this stage it is just a homogeneous compact body with mass M "concentrated" at the origin. The radial extension of the massive body does not matter as long as it is smaller than or equal to r. This is so, because if r is smaller than the radial extension of the large body of mass M, then some amount of mass will not be taken into account when calculating the force of attraction at radius r. When assuming r to be the radial extension of the body of mass M, the following equations express the force acting on the surface of the body, i.e. at radius r. Eqs. 85 and 86 describe the motion of a body in a central static field. The gravitational acceleration is as in (83), so that

$$c^2\left(1-\frac{GM}{rc^2}\right)^{-2}\left(-\frac{GM}{r^2c^2}+3\frac{G^2M^2}{c^4r^3}-2\frac{G^3M^3}{c^6r^4}\right).$$

$F(r)$ is the radial force acting on a unit mass (unit $[N]$). Let us evaluate that force. The central mass M is supposed to be concentrated at the origin or at least on a radius smaller than r. We choose the unit system $\frac{GM}{c^2}=1$ for brevity and clarity. Then,

$$F(r)=(1-\frac{1}{r})^{-2}(-\frac{1}{r^2}+\frac{3}{r^3}-\frac{2}{r^4}) \tag{128}$$

has two singularities 1 at $r=0$ and 0 at r=2. Between r=0 and r= $\frac{GM}{c^2}$ the force is always attractive ($F<0$, Fig. 10). The region from r=0 to $r=\frac{GM}{c^2}$ is attractive, with infinite attraction for $r=0$ and

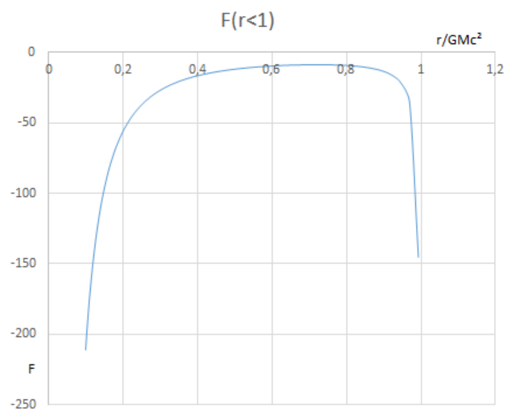

Figure 10. F for $r<GM/c^2$

$r=\frac{GM}{c^2}$. Between $r=\frac{GM}{c^2}$ and $r=2\frac{GM}{c^2}$ the force is repulsive ($F>0$) and is attractive again for $r>2\frac{GM}{c^2}$. The force then follows a $\frac{1}{r^2}$ law for large r. The point $2\frac{GM}{c^2}$ is stable in equilibrium with

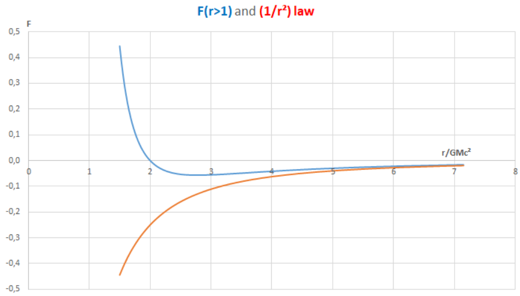

Figure 11. F for $r > GM/c^2$

zero force, and the corresponding radius is called the limit radius R_L. Note that it is equal to the Schwarzschild horizon in GR. *A black hole can never shrink to a null radius*: An infinite repulsive barrier at $r = \frac{GM}{c^2}$ prevents this collapse to happen and R_L represents the stable size of a non-rotating black hole. At R_L, the surface of the black hole is in the equilibrium and no force is acting on the surface. The force (83) derives from the following potential $\Phi(r)$, for $r > \frac{GM}{c^2}$ and set to zero at r_∞

$$\Phi(r) = c^2 \left(\frac{2GM}{rc^2} + \ln\left(1 - \frac{GM}{rc^2}\right)\right) [\frac{m^2}{s^2}]. \tag{129}$$

Which can be developed in the series:

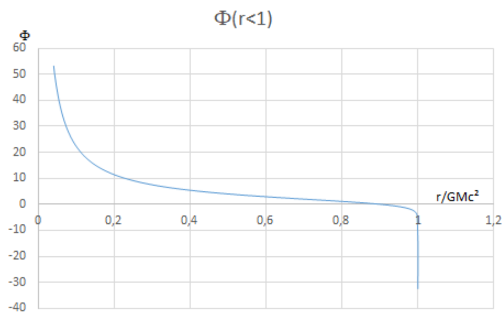

Figure 12. Φ for $r < GM/c^2$

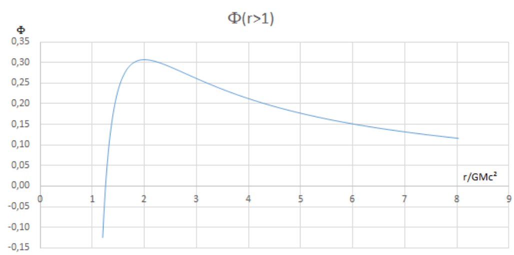

Figure 13. Φ for $r > GM/c^2$

$$\Phi(r) = \frac{GM}{r} - \frac{G^2M^2}{2r^2c^2} - \frac{G^3M^3}{3r^3c^4} - \dots . \tag{130}$$

This series is the sum of an attractive potential $\frac{GM}{r}$ and a repulsive potential equal to $-\frac{G^2M^2}{2r^2c^2} - \frac{G^3M^3}{3r^3c^4} - \dots$. The latter repulsive term could be seen as a graviton-graviton interaction term:Iit is negligible for large distance r but predominant at short distances corresponding to the Schwarzschild radius. It has a maximum of $c^2(1 + \ln(1/2)) \simeq 0.307c^2$ at R_L.

The escape of light from a "black hole"

In Fig. 14 we plot $\check{}\Phi$ for a more intuitive understanding. We follow a classical approach equaling potential and kinetic energies. We have for a unit mass m

$$1/2\, mv^2 = mc^2 \left(\frac{2GM}{rc^2} + \ln\left(1 - \frac{GM}{rc^2}\right)\right).$$

Dividing by m and equalling v to c gives rise to

$$1/2 = \left(\frac{2GM}{rc^2} + \ln\left(1 - \frac{GM}{rc^2}\right)\right).$$

There is no solution to this equation, the kinetic energy is always higher than the potential well, so

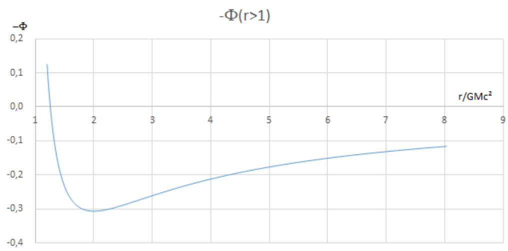

Figure 14. $-\Phi$ for $r > GM/c^2$

light can always escape from the "black hole". We have to find a new name for this class of bodies having a radius close or equal to R_L: They're not holes and they are not black either. Let us call them CORE. Quasars could be such cores presenting red shifts and pulsations, as will be shown in the next sections. The core radius of stability R_L is equal to the Schwarzschild radius of GR. The core sits there in stable equilibrium between expansion and contraction forces. If there was no repulsive terms, $-\frac{G^2M^2}{2r^2c^2} - \frac{G^3M^3}{3r^3c^4} - \dots$, in the potential, then a solution would be possible where no light can escape at a horizon radius equal to the Schwarzschild radius and a gravitational singularity would appear.

10. Red shift of a core

The core is still considered as a non-rotating mass M. From (1) we obtain (by taking the potential Φ into account)

$$\nu_1 \left(1 - \frac{\Phi_1}{c^2}\right) = \nu_2 \left(1 - \frac{\Phi_2}{c^2}\right).$$

If the position 1 is at infinite distance $r = \infty$, then $\Phi_1 = 0$ At position 2, $r = R_L$ and $\Phi_2 = 0.307c^2$ and then $\nu_1 = \nu_2(1 - 0.307) = 0.693\, \nu_2$. The frequency of a spectral line emitted from a point at the surface R_L of a core will be perceived from an infinite distance at 0.693 times the frequency due to time slowdown at the surface of the core. But if we take into account the potential well of $0.307\,c^2$, the emitted frequency would decrease further by a factor of 0.614 since $E = h\nu$ and E is reduced by a factor $\frac{0.307c^2}{((c^2)/2)}$. The total frequency shift factor is the 0.693 times $(1 - 0.614)$ which is equal to 0.268.

11. Pulsation of a core

On the surface of the core, a mass is at equilibrium but can also oscillate radially around the equilibrium point R_L. Let us look at its first mode of oscillation: For a unit mass $m = 1$ on the surface of the core, the oscillation frequency ω is $\sqrt{\frac{k}{m}} = \sqrt{k}$, and

$$k = \frac{dF}{dr}|_{R_L} = -\frac{c^6}{4G^2M^2} \tag{131}$$

$$\sqrt{|k|} = \frac{c^3}{2GM}$$

Thus, $\omega = \frac{c^3}{2GM}$ for a core of mass M. For instance a non-rotating core of mass = 1000 times the sun mass would pulsate on its first mode at $\omega = 101\,rad/s = 16Hz$.

12. Expansion of the universe

The estimated mass of the known universe is in a range $1.7\,10^{52}$ to $1.7\,10^{54}\,kg$. Let us calculate the R_L of the universe.

$$R_L = \frac{2GM}{c^2} = 2.5\,10^{25} \text{ to } 2.5\,10^{27}\, m.$$

The estimated radius of the universe according to the standard cosmological model is $46\,10^9$ light years = $4.2\,10^{26}\,m$. So the estimated radius of the universe is in the same range of magnitude as its R_L radius and it could even be very close to its R_L radius! And the universe could then have some properties of a core. Figure 15 represents $-\Phi$ of the universe between 0.7 and $2\,R_L$.

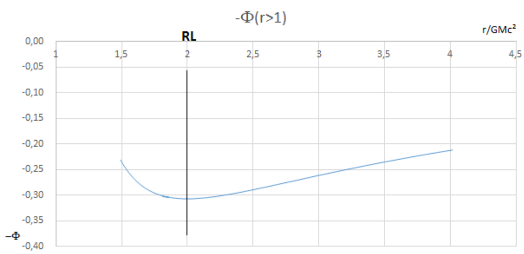

Figure 15. $-\Phi$ for $r > GM/c^2$

If considered as a non rotating core, the universe would pulsate around its R_L size, at a frequency $\frac{c^3}{2GM}$ or a period $\frac{4\pi c^3}{2GM} = 278\,10^9$ years, considering $R_L = 4.2\,10^{26} m = \frac{2GM}{c^2}$ (point 2 of Fig 16).

If the universe is now in an expansion phase, this would mean that its size is presently lower than its horizon R_L.

If the universe had begun at a size $< 1.27\,R_L/2$, it would have enough potential energy to expand to an infinite radius if there is no energy loss during that expansion. Otherwise the universe will oscillate or fluctuate around its R_L radius, where Φ is maximum. This resembles the A. D. Sakharov's concept of the fluctuating or oscillating universe (Al'tshuler 1991).

13. Acceleration of the expansion of the universe

Let us consider a body of mass m situated on the rim R_L of the universe, the force acting on this body is (83)

$$F = m\,c^2 \frac{1}{\left(1 - \frac{GM}{c^2 r}\right)^2} \left(\frac{-GM}{c^2 r^2} + 3\frac{G^2M^2}{c^4 r^3} - 2\frac{G^3M^3}{c^6 r^4} \right).$$

The equation of motion is

$$m\,\ddot{r} - c^2 \frac{1}{\left(1 - \frac{GM}{c^2 r}\right)^2} \left(\frac{-GM}{c^2 r^2} + 3\frac{G^2M^2}{c^4 r^3} - 2\frac{G^3M^3}{c^6 r^4} \right) m = 0$$

with the potential given by

$$\Phi(r) = c^2 \left(\frac{2GM}{rc^2} + \ln\left(1 - \frac{GM}{rc^2}\right)\right).$$

Again, we represent $-\Phi$ on the graphs, to make the presentation more intuitive. For small motion

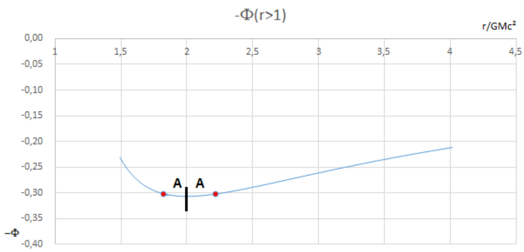

Figure 16. Amplitude A

around the value R_L, we approximate $F(r)$ by an harmonic force $F(r) = k(r - RL)$ with $k = \frac{c^6}{4G^2M^2}$ (eq. 131) and with $r = R_L$ at $t = 0$ we get:

$$(r - R_L) = A \sin\left(\frac{c^3}{GM} t\right)$$

A being the maximal fluctuation in size of the universe, and

$$(r - \dot{R_L}) = A \frac{c^3}{GM} \cos\left(\frac{c^3}{GM} t\right)$$

is the expansion rate of the universe, and

$$(r - \ddot{R_L}) = -A \frac{c^6}{G^2M^2} \sin\left(\frac{c^3}{GM} t\right)$$

is the acceleration of the expansion of the universe. When $r < R_L$, the expansion and the acceleration are both positive, but with a negative rate of the acceleration. This is a common feature of periodic motion. When the radius of the universe will reach its horizon R_L the expansion will continue but at a decelerating rate until the universe reaches its maximum size $R_L + A$ and stops expanding. Then the inverse movement will take place.

A and t are two unknowns which can be determined by the values of the expansion rate (Hubble-Lemaître constant = 70 km/s/Mps) and the value of the acceleration of the universe expansion.

14. Conclusion

The present model for gravitation is not equivalent to general relativity in that the field Γ determines the geometry and plays a central role. The existence of a short range repulsive potential leads to very interesting results, especially with regard to black holes which should not exist but instead would be "cores" or "compact bodies" that have a finite size. The current that generates the gravitational forces is the Energy-Impulsion tensor and this is a natural consequence of the invariance of the action under the group of translations in space-time. The repulsive $-\frac{G^2M^2}{r^2}$ potential term is absent in general relativity. For that reason and in order to explain the acceleration of the expansion of the universe, the influence of a hypothetical 'dark energy' was invoked in GR. *Our model does not need 'dark energy' to explain the acceleration of the expansion of the universe.* The repulsive $-\frac{G^2M^2}{r^2}$ potential tends to open the orbit of Mercury and this contributes to fix the advance the perihelion to $42.3''$ per century. This is in good agreement with the measured value of $43''$. The expansion of the universe could be a consequence of the universe being considered as a core with a natural pulsation frequency of one cycle per $278\ 10^9$ years. As such the universe would radially oscillate around an equilibrium point instead of being originated from a 'Big Bang". The speed distribution in rotating galaxies arms could also be calculated in the new theoretical model, possibly taking into account a "Lorentz" force. What's more, we can show that this "Lorentz" force acts in the right centripetal direction without maybe having to rely on "dark matter" to do the job. Quantum gravity should have the massless spin 2 graviton for the propagator of the interaction and this quantification could be the subject of further work. Rotating cores are also a topic for further study.

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