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Logemes and Their Homotopy-Theoretic Foundations

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Abstract

We introduce *logemes*, consistent fragments of reasoning closed under at least one inference rule, as foundational units for different logics. Unlike full logical theories, logemes need not be axiomatized or algebraically structured; instead, they are evaluated via their associated Lindenbaum–Tarski quotients, interpreted as spaces presenting partially ordered sets. We propose that two logemes are homotopy identical precisely when their posetal semantics are homotopy equivalent. This criterion, grounded in the Univalence Axiom of Homotopy Type Theory, allows us to formalize diagrammatic reasoning and compare ancient logical traditions, such as Stoic and Yogācāra, on purely mathematical grounds. We show that both traditions instantiate the same homotopy type of poset, confirming their logical identity despite historical separation.

Keywords: simplicial complex, homotopy type, univalent foundations

1. Introduction

From the standpoint of modern mathematical logic, pre-20th-century logical systems, such as Stoic, Epicurean, Aristotelian, and Buddhist, are not theories in the formal sense, since they lack axiomatizations, completeness results, and algebraic semantics. Nevertheless, they contain coherent doctrines of inference, introducing explicit rules for deriving conclusions from premises.

We propose that such doctrines are best understood as collections of *logemes* – minimal, usable fragments of reasoning. A logeme is not a theory but a *reasoning token*, i.e., a finite, consistent set of formulas on which at least one inference rule can be applied (e.g., *modus ponens* on $\{p, p \Rightarrow q, q\}$). This reflects everyday practice: legal, medical, or scientific argumentation rarely invokes full logical systems, but relies instead on reusable, local inferential patterns. Hence, the central thesis of this paper is as follows: *the history of logic prior to Frege and Russell is the history of logemes, not of logical theories.*

To formalize this approach, we equip logemes with semantic content via Lindenbaum–Tarski quotients, then classify them up to *homotopy equivalence* of their induced posets. This approach, inspired by univalent foundations Univalent Foundations Program 2013, treats logically equivalent reasoning patterns as *homotopy identical* when their diagrammatic realizations share the same homotopy type.

The paper proceeds as follows. In section 2, we define logemes and their semantics. Section 3 surveys tools for computing homotopy types of posets. Section 4 establishes the homotopy-theoretic identification principle. In section 5, we apply the framework to Stoic and Yogācāra logemes, proving their identity. We conclude with historical and methodological implications.

2. Logemes and Their Semantics

Let T be a (classical or intuitionistic) propositional logic, with algebra of formulas A_T. For any formula p, define its *equivalence class*:

$$\|p\|_T := \{q: \ \vdash_T \ p \equiv q\}.$$

Let $\equiv_T$ be the congruence induced by $\equiv$. The *Lindenbaum-Tarski algebra* of T is the quotient $A_T/{\equiv_T}$. When T is classical, this is a Boolean algebra; for intuitionistic T, a Heyting algebra.

Definition 1 (Logeme). *A finite set of formulas $F \subseteq A_T$ is a* logeme *iff F is* consistent*: $\nvdash_T \bigwedge F \Rightarrow \bot$. Furthermore, this F is not trivial iff it is inferentially closed under at least one rule: there exists an inference rule R of T and a substitution σ such that the premises of $R\sigma$ lie in F, and its conclusion is entailed by F (not necessarily in F itself).*

The set $F = \{p, p \Rightarrow q\}$ is a logeme: it is consistent (in any nontrivial T), and *modus ponens* applies to yield q. The singleton $\{p\}$ is trivial, as no rule applies; and $\{p, \neg p\}$ is inconsistent, hence excluded.

Given a logeme F, we consider the subalgebra $A_F \subseteq A_T$ generated by F, and its quotient $A_F/{\equiv_T}$. However, for historical logics, we cannot assume a background theory T. We thus define semantics intrinsically:

Definition 2 (Meaningful Logeme). *A logeme F is* meaningful *if the Lindenbaum quotient $A_F/{\equiv}$ (for the fragment it generates) carries a partial order $\leq$ such that:*

$$\|p \Rightarrow q\| = 1 \quad \textit{iff} \quad \|p\| \leq \|q\|.$$

In this case, $A_F/{\equiv}$ is a poset*; it is a* lattice *if all binary suprema and infima exist.*

Proposition 1. *The logeme $F = \{(p\,\&\,(p \Rightarrow q)) \Rightarrow q\}$ is meaningful for the 2-element chain $\mathbf{2} = \{0 < 1\}$ under Boolean interpretation.*

Proof. Indeed:

$$\begin{aligned}\|(p\,\&\,(p \Rightarrow q)) \Rightarrow q\| &= \|\neg(p\,\&\,(\neg p \vee q)) \vee q\| \\ &= \|(\neg p \vee \neg(\neg p \vee q)) \vee q\| \\ &= \|((\neg p \vee q) \vee \neg(\neg p \vee q))\| = 1.\end{aligned}$$

Hence, the formula evaluates to 1 (top element), and *modus ponens* holds in $\mathbf{2}$. □

Thus, logical reasoning reduces to operating on *individual meaningful logemes*, each tied to a class of posets. The space of all such logemes vastly exceeds the space of formal logical systems.

3. Computational Tools for Homotopy Types of Posets

To classify logemes via homotopy, we require effective methods to compute or simplify the homotopy type of the order complex $|\Delta(P)|$ of a finite poset P. Below we recall and elaborate the main combinatorial-topological tools.

Definition 3 (Order Complex). *Let P be a (finite) partially ordered set. The* order complex *of P, denoted $\Delta(P)$, is the abstract simplicial complex defined by:*

$$\Delta(P) := \Big\{\{x_0, x_1, \ldots, x_k\} \subseteq P: \ k \geq 0, \ x_0 < x_1 < \cdots < x_k \text{ in } P\Big\}.$$

Each such chain is a k-simplex; the face relation is given by inclusion of subchains.

The geometric realization $|\Delta(P)|$ is the topological space obtained by assigning to each k-simplex $\{x_0 < \cdots < x_k\}$ a standard geometric k-simplex in $\mathbb{R}^{k+1}$ and gluing them along common faces. This yields a finite CW-complex, canonically associated to P.

Remark 1. *The construction is functorial: an order-preserving map* $f\colon P \to Q$ *induces a simplicial map* $\Delta(f)\colon \Delta(P) \to \Delta(Q)$, *and hence a continuous map* $|\Delta(f)|\colon |\Delta(P)| \to |\Delta(Q)|$.

The homotopy type of $|\Delta(P)|$ is an invariant of P; it captures global connectivity properties (e.g., presence of "holes") while ignoring local combinatorial details.

Example 1. *Let us consider some easy cases:*

1. *If* $P = \{a\}$ *is a singleton,* $\Delta(P) = \{\{a\}\}$ *and* $|\Delta(P)|$ *is a point.*
2. *If* $P = \{a, b\}$ *with* $a \parallel b$ *(incomparable), then* $\Delta(P) = \{\{a\}, \{b\}\}$ *(no 1-simplex), so* $|\Delta(P)|$ *is a discrete 2-point space, homotopy equivalent to* S^0.
3. *If* $P = \{a < b\}$, *then* $\Delta(P) = \{\{a\}, \{b\}, \{a, b\}\}$, *and* $|\Delta(P)|$ *is a closed interval and then it is contractible.*

A powerful simplification technique for finite posets is *core reduction*, introduced by Stong 1966 and later refined in the homotopy-theoretic context by Raptis 2010.

Definition 4 (Upbeat and Downbeat Points)**.** *Let P be a finite poset and* $x \in P$.

- *x is an* upbeat point *if the set* $U(x) := \{y \in P\colon y > x\}$ *is nonempty and has a* least element, *denoted* $x^{\uparrow}$.
- *x is a* downbeat point *if the set* $L(x) := \{y \in P\colon y < x\}$ *is nonempty and has a* greatest element, *denoted* $x^{\downarrow}$.

Equivalently, x is upbeat iff there exists $x^{\uparrow} > x$ *such that for all* $y > x$, $y \geq x^{\uparrow}$; *similarly for downbeat.*

Intuitively, an upbeat point x is "redundant": it sits directly below a unique minimal element above it, so removing x and identifying it with $x^{\uparrow}$ does not change the homotopy type.

Theorem 1 (Core Reduction: Stong 1966; Raptis 2010)**.** *Let P be a finite* T_0*-poset (i.e. the specialization preorder is antisymmetric – it is automatically true for posets). Then*

1. *There exists a unique subposet* core$(P) \subseteq P$, *called the* core *of P, such that:*
   - core(P) *contains no upbeat or downbeat points;*
   - core(P) *is a* strong deformation retract *of P in the Alexandrov topology (hence* $|\Delta(P)| \simeq |\Delta(\text{core}(P))|$*).*
2. core(P) *is obtained recursively: repeatedly delete any upbeat or downbeat point until none remain; the result is independent of deletion order.*
3. *For finite posets P and Q, we have*

$$|\Delta(P)| \simeq |\Delta(Q)| \iff \text{core}(P) \cong \text{core}(Q) \text{ as posets.}$$

Example 2. *Consider the following 4-element poset:*

$$P = \{a, b, c, d\}, \quad a < c,\ b < d,\ a \parallel b,\ c \parallel d,\ a \parallel d,\ b \parallel c.$$

No element has a unique cover above or below, e.g., $U(a) = \{c\}$ *has least element c, but* $U(a) = \{c\}$ *is a singleton, so c is its minimum. Thus a is upbeat, with* $a^{\uparrow} = c$. *Similarly b is upbeat* ($b^{\uparrow} = d$), *c is downbeat* ($c^{\downarrow} = a$), *d is downbeat* ($d^{\downarrow} = b$). *But core reduction forbids removing both endpoints of a covering pair simultaneously. In fact, the core is obtained by removing one pair, e.g., delete a and b; the remaining poset* $\{c, d\}$ *with* $c \parallel d$ *is discrete 2-point. Thus, its order complex is* S^0.

Example 3. *Consider another 4-element poset:*

$$P = \{0, a, b, 1\}, \quad 0 < a < 1,\ 0 < b < 1,\ a \parallel b.$$

This is the Boolean lattice $\mathbf{2}^2$. *Now:*

- *a: $U(a) = \{1\}$, then it is upbeat ($a^{\uparrow} = 1$);*
- *b: $U(b) = \{1\}$, then it is upbeat;*
- *1: $L(1) = \{a, b\}$ has no greatest element (since $a \parallel b$); therefore, it is not downbeat;*
- *0: $U(0) = \{a, b\}$ has no least element; therefore, it is not upbeat.*

Removing a and b leaves $\{0, 1\}$ with $0 < 1$, but it is a contractible chain.

Example 4. *A new 4-element poset:*

$$P = \{-1, a, b, +1\}, \quad -1 < a < +1,\ -1 < b < +1,\ a \parallel b.$$

Let its order complex be as follows:

$$\Delta(P) = \{\{-1\}, \{a\}, \{b\}, \{+1\}, \{-1, a\}, \{-1, b\}, \{a, +1\}, \{b, +1\}\}.$$

Then, geometrically, this is a 1-dimensional simplicial complex consisting of two edges sharing endpoints, but it is a circle S^1. Indeed,

$$|\Delta(P)| \cong S^1.$$

Beyond core reduction, several other powerful techniques exist:

1. **Cross-cuts** (Lakser 1971). Let L be a bounded lattice ($0 = \min L$, $1 = \max L$). A *cross-cut* is a subset $X \subseteq L \setminus \{0, 1\}$ such that:

   (i) X is an antichain (no two elements comparable);
   (ii) Every maximal chain in L contains exactly one element of X.

   Then the inclusion-induced map $|\Delta(X)| \hookrightarrow |\Delta(L)|$ is a homotopy equivalence. In particular, if X is discrete with k elements, $|\Delta(L)| \simeq \bigvee^{k-1} S^0$.
2. **Contractible Carriers** (Walker 1981). Let K be a simplicial complex and Y a topological space. A *carrier* is a function $C\colon \{\text{simplices of } K\} \to \{\text{subspaces of } Y\}$ such that $\sigma \subseteq \tau \Rightarrow C(\sigma) \subseteq C(\tau)$. It is *contractible* if each $C(\sigma)$ is contractible. Then:

   (a) There exists a continuous map $f : |K| \to Y$ with $f(|\sigma|) \subseteq C(\sigma)$ for all σ (*f is carried by C*);
   (b) Any two such maps are homotopic.

   This is used to construct homotopy equivalences combinatorially.
3. **Order Homology**. For $n \geq 0$, define the *order homology groups* of P by:

   $$H_n(P; \mathbb{Z}) := H_n^{\text{sing}}(|\Delta(P)|; \mathbb{Z}),$$

   the singular homology of the geometric realization. Equivalently, one may use simplicial homology of $\Delta(P)$. These are homotopy invariants:

   $$P \simeq Q \Rightarrow H_n(P) \cong H_n(Q)\ \forall n.$$

   For instance:

   $$H_0(P) \cong \mathbb{Z}^c, \quad c \text{ is a number of connected components of } |\Delta(P)|,$$

   and $H_1(P) \cong \mathbb{Z}^r$ detects the number r of independent 1-dimensional loops.

These tools enable a systematic classification of logeme semantics. Given a logeme F, one proceeds step by step as follows:

1. Compute its associated poset presentation $P_F := A_F/\equiv$.

2. Reduce P_F to its core, core(P_F), by iteratively stripping away all dismantlable or contractible elements. This yields a canonical representative in the homotopy class of P_F.
3. Compute the (co)homology groups $H_*(P_F)$, or alternatively recognize P_F (or core(P_F)) as a known combinatorial poset or simplicial shape (e.g. a diamond $\mathbf{2}^2$, a Boolean lattice $\mathbf{2}^n$, a sphere, or a bouquet of circles).
4. From these invariants determine the homotopy type of the semantic space carried by F.

Two logemes F and G are identified precisely when their associated invariants coincide; that is,

$$\mathrm{core}(P_F) \simeq \mathrm{core}(P_G) \quad \text{and} \quad H_*(P_F) \cong H_*(P_G),$$

ensuring that they define the same semantic homotopy type. This procedure yields a complete and structurally transparent classification of logemes.

4. Homotopy Identification of Logemes

To systematize logemes in a manner compatible with both historical reasoning and modern mathematics, we work within the framework of *univalent foundations* (Univalent Foundations Program 2013). At its core lies the *Univalence Axiom*, which reinterprets equality of mathematical objects as equivalence:

Axiom 1 (Univalence Axiom). *For any two types (spaces) X and Y, the identity type $X = Y$ is equivalent to the type of homotopy equivalences between them:*

$$(X = Y) \simeq (X \simeq Y).$$

Let F be a meaningful logeme (cf. definition 2). Its Lindenbaum quotient $A_F/{\equiv}$ carries a natural partial order $\leq$, making it a finite poset, denoted

$$P_F := A_F/{\equiv}.$$

As explained in section 3, every finite poset gives rise to a *simplicial complex* $\Delta(P_F)$, or its *order complex*, whose geometric realization $|\Delta(P_F)|$ is a finite CW-complex. This space encodes the global "shape" of the logeme: connected components correspond to independent reasoning fragments, loops to cyclic dependencies, and higher holes to unresolvable contradictions.

We thus treat P_F not merely as a combinatorial object, but as a space via the composite functor

$$F \longmapsto P_F \longmapsto \Delta(P_F) \longmapsto |\Delta(P_F)|.$$

The Univalence Axiom motivates two related but distinct relations on logemes:

Definition 5 (Strict Identity of Logemes). *Two logemes F and F' are* strictly identical*, written $F = F'$, iff their order complexes are the same:*

$$F = F' \quad \Longleftrightarrow \quad \Delta(P_F) = \Delta(P_{F'}).$$

Strict identity means that combinatorically both logemes are identical. More useful is the univalent notion:

Definition 6 (Homotopy Identity of Logemes). *Two logemes F and F' are* homotopy identical *(in the univalent sense), written $F \simeq F'$, iff their realizations are homotopy equivalent:*

$$F \simeq F' \quad \Longleftrightarrow \quad |\Delta(P_F)| \simeq |\Delta(P_{F'})|.$$

Note that under the Univalence Axiom (axiom 1), this is not merely a definition, since it is a justified identification: the identity type $F \simeq F'$ is the type of homotopy equivalences between $|\Delta(P_F)|$ and $|\Delta(P_{F'})|$.

The notation $|\Delta(P_F)|$ denotes the geometric realization of the order complex of the poset P_F. In the present framework, this space admits a natural interpretation as a *logic diagram* for the formulas generated by the logeme F.

More explicitly, the simplices of $\Delta(P_F)$ correspond to chains of entailments or dependency relations among the formulas encoded in F, while their geometric realization $|\Delta(P_F)|$ provides a continuous topological space in which these logical relations are represented as higher-dimensional cells. Thus, $|\Delta(P_F)|$ serves as a faithful topological model of the inferential structure of F, capturing not only the individual formulas but also the ways in which they combine, branch, or form cycles within the logical system. It provides a rigorous mathematical framework for *logic diagrams* as a bona fide logical discipline in its own right, distinct from syntactic proof theory or model-theoretic semantics (Anger et al. 2022, Schumann and Lemanski 2022) – by formalizing their structural invariants through homotopy-theoretic and order-theoretic methods. Specifically, diagrams are no longer treated as heuristic illustrations, but as *combinatorial models* whose inferential content is encoded in the homotopy type of their associated posets (or, equivalently, the geometric realization of their order complexes). This approach validates the five "dogmas" challenged in Anger et al. 2022, e.g., the belief that diagrams are inherently ambiguous, non-compositional, or incapable of expressing generality, by demonstrating that:

(i) *Syntax* can be captured via formulas from F;
(ii) *Semantics* arises from the simplicial complex $\Delta(P_F)$ for the poset P_F obtained based on the formulas of F;
(iii) *Inference rules* correspond to homotopy-preserving transformations (e.g., core reduction, barycentric subdivision, or strong collapses);
(iv) *Soundness and completeness* can be formulated as homotopy equivalences between diagram spaces and algebraic models (e.g., Boolean or Heyting algebras);
(v) *Diagrammatic equivalence* coincides with univalent identity: two diagrams represent the same reasoning pattern iff their realizations are homotopy equivalent.

Thus, the study of logical diagrams is elevated from a pedagogical aid to a formal branch of logic, what one may call *homotopy theory of logic diagrams* with its own syntax, semantics, and proof theory grounded in univalent foundations. As we can see, univalent foundations provide not only a formal framework, but a *conceptual clarification*, for which logical reasoning is not about manipulating symbols in a fixed syntax, but about navigating a space of homotopy types, where identity is sameness of shape, and inference is continuous transformation.

5. Case Study: Stoic and Yogācāra Logemes

We now apply the univalent framework to two historically and geographically distinct traditions: Hellenistic Stoicism (3rd c. BCE) and Indian Yogācāra Buddhism (5th c. CE). Despite a millennium of separation and no known direct textual transmission, their core inferential systems exhibit striking structural parallels. We show that this is homotopy equivalence.

The Stoics codified five *anapodeiktoi sullogismoi* (indemonstrable syllogisms), see Bobzien 1999, captured by the logeme:

$$F_{\text{Stoic}} = \left\{ \begin{array}{ll} \text{(MP)} & ((p \Rightarrow q) \,\&\, p) \Rightarrow q, \\ \text{(MT)} & ((p \Rightarrow q) \,\&\, \neg q) \Rightarrow \neg p, \\ \text{(MPT)} & (\neg(p \,\&\, \neg q) \,\&\, p) \Rightarrow \neg\neg q, \\ (\text{MTP}_1) & ((\neg p \oplus q) \,\&\, \neg p) \Rightarrow \neg q, \\ (\text{MTP}_2) & ((\neg p \oplus q) \,\&\, \neg\neg p) \Rightarrow q. \end{array} \right\}. \quad (1)$$

These correspond respectively to *modus ponens* (MP), *modus tollens* (MT), *modus ponendo tollens* (MPT), and two variants of *modus tollendo ponens* (MTP_1 and MTP_2).

Their semantics is given by the square of opposition in Figure 1, interpreted as a poset P_{Stoic}, in which $||p|| :=$ "day", $||q|| :=$ "light", $||\neg p|| :=$ "dark", $||\neg q|| :=$ "night".

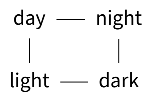


Figure 1. The Stoic square of opposition. Bottom edge is *contradictory* (not subcontrary).

In this poset P_{Stoic}, we know that $||q \oplus \neg p|| = 1$. Then

$$\Delta(P_{\text{Stoic}}) = [\{||p||, ||q||, ||q \oplus \neg p||\}, \{||\neg q||, ||\neg p||, ||q \oplus \neg p||\}],$$

i.e., $\Delta(P_{\text{Stoic}})$ consists of two 2-simplices, see Figure 2. It is worth noting that $|\Delta(P_{\text{Stoic}})|$ is contractible (collapsed to a point).

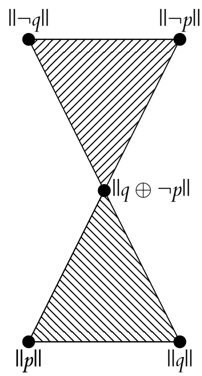


Figure 2. The logic diagram $|\Delta(P_{\text{Stoic}})|$ for the Stoic square of opposition, see Figure 1.

Dharmakīrtī's system (Dreyfus 1997), systematized in the *Nyāyabindu*, employs 13 context-sensitive inference rules, categorized by token type (analytic, causal, genus, etc.) and compositionally extended (e.g., *composite modus tollendo ponens*). So, the Yogācāra logeme F_{Yogacara} is defined as follows:

(i) (*MP*, analytic token) $((p \Rightarrow q)\&p) \Rightarrow q$,
(ii) (*MP*, causal token) $((p \Rightarrow q)\&p) \Rightarrow q$,
(iii) (*MT*, analytic token) $((p \Rightarrow q)\&\neg q) \Rightarrow \neg p$,
(iv) (*MT*, causal token) $((p \Rightarrow q)\&\neg q) \Rightarrow \neg p$,
(v) (*MT*, genus) $((p \Rightarrow q)\&\neg q) \Rightarrow \neg p$,

(vi) (composite MTP_1) $((p \Rightarrow q)\&(q \oplus \neg p)\&p) \Rightarrow \neg\neg p$,
(vii) (composite MTP_1) $((q \oplus \neg p)\&p\&((p \Rightarrow r)\&(r \Rightarrow q))) \Rightarrow \neg\neg p$,
(viii) (composite MPT) $(\neg(q\&\neg p)\&p\&(p \Rightarrow q)) \Rightarrow \neg\neg p$,
(ix) (MPT, cause) $(\neg(q\&\neg p)\&q) \Rightarrow \neg\neg p$,
(x) (composite MTP_1) $((q \oplus \neg p)\&p\&(p \Rightarrow q)) \Rightarrow \neg\neg p$,
(xi) (MT, cause) $((p \Rightarrow q)\&\neg q) \Rightarrow \neg p$,
(xii) (composite MTP_2) $((q \oplus \neg p)\&p\&(p \Rightarrow \neg q)) \Rightarrow \neg p$,
(xiii) (MPT, token) $(\neg(q\&\neg p)\&q) \Rightarrow \neg\neg p$.

These rules are contextually enriched variants of the Stoic indemonstrables: *modus ponens* (MP), *modus tollens* (MT), *modus ponendo tollens* (MPT), and two versions *modus tollendo ponens* (MTP_1 and MTP_2).

Both logemes (Stoic and Yogācāra) are verified on the same poset. Indeed, the Yogācāra semantics is given by the square of opposition in Figure 3, interpreted as a poset P_{Yogacara}, in which $\|p\| :=$ "fire", $\|q\| :=$ "warmth", $\|\neg p\| :=$ "cold", $\|\neg q\| :=$ "ice".

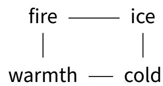


Figure 3. The Yogācāra square of opposition. Bottom edge is also *contradictory* (not subcontrary).

Yogācāra logic, like its Stoic counterpart, is grounded in a theory of *sign-inference* (*anumāna*). A phenomenon q (the *signified*, *sādhya*) is inferred from its *sign* or *token* (*liṅga*, p), provided p is invariably connected with q (the *vyāpti* condition).

Proposition 2. *Under the above interpretation, $P_{\text{Yogacara}} \cong P_{\text{Stoic}}$ as posets. Consequently:*

$$\Delta(P_{\text{Yogacara}}) \cong \Delta(P_{\text{Stoic}}).$$

□

Hence, we conclude:

Theorem 2 (Stoic–Yogācāra Logeme Identity)**.** *The Stoic logeme F_{Stoic} and the Yogācāra logeme F_{Yogacara} are identical:*

$$F_{\text{Stoic}} = F_{\text{Yogacara}}.$$

□

In contrast, the *Aristotelian* syllogistic logeme $F_{\text{Aristotelian}}$, modeled on the classical square of opposition (with subcontrary bottom) is not homotopy equivalent to Stoic or Yogācāra logemes: $F_{\text{Aristotelian}} \not\simeq F_{\text{Stoic}}$.

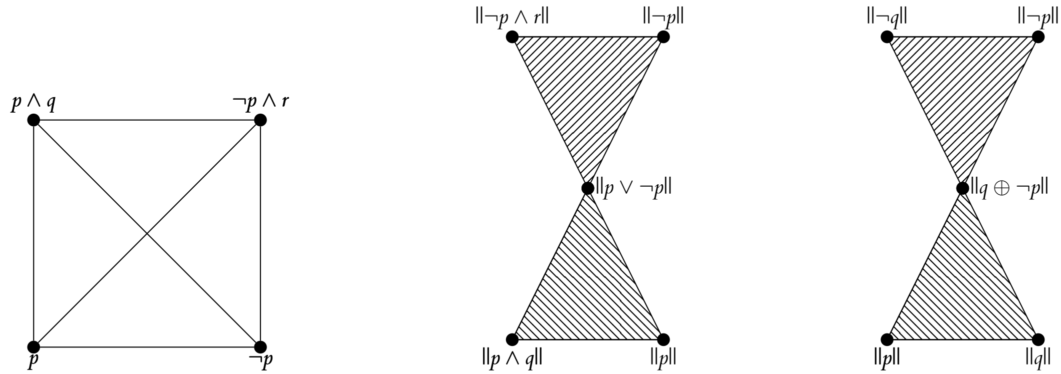
Historically, the earliest attested logemes appear in Mesopotamian legal and divinatory texts (2nd millennium BCE), explicitly deploying MP and MT (Schumann 2023). No such reasoning fragments are found in contemporaneous cultures lacking Sumerian–Akkadian influence. The homotopy identity $F_{\text{Stoic}} = F_{\text{Yogacara}}$ thus supports a diffusionist hypothesis: Hellenistic logical doctrine likely influenced early Indian Buddhist epistemology along Silk Road intellectual networks. This case exemplifies how univalent foundations can be applied to two historically distinct traditions.

Remarkably, the same homotopy type arises even within some fragments of *classical propositional logic* (CPL):

Example 5 (Classical Realization)**.** *Consider the* CPL*-logeme:*

$$F_{\text{CPL}} = \Big\{(p \wedge q) \Rightarrow p,\ (\neg p \wedge r) \Rightarrow \neg p,\ p \oplus \neg p,\ \neg\big((p \wedge q) \wedge (\neg p \wedge r)\big)\Big\}.$$

Its semantic poset P_{CPL} is isomorphic to P_{Stoic}, and $\Delta(P_{\mathsf{CPL}}) \cong \Delta(P_{\mathsf{Stoic}}) \cong \Delta(P_{\mathsf{Yogacara}})$, see Figure 4. Thus, the Stoic, Yogācāra, and even a fragment of classical logic converge on the same order complex *– a two 2-simplices connected at one point.*



(a) The Stoic square of opposition for F_{CPL} (b) $|\Delta(P_{\mathsf{CPL}})|$ (c) $|\Delta(P_{\mathsf{Stoic}})| \cong |\Delta(P_{\mathsf{Yogacara}})|$

Figure 4. (a) The Stoic square of opposition for the logeme F_{CPL}; (b) the simplicial complex of the Stoic square of opposition for the logeme F_{CPL}; (c) the simplicial complex of the Stoic square of opposition for the logeme F_{Stoic} of the Stoics as well as for the logeme F_{Yogacara} of the Yogācārins.

6. Conclusion

We have formalized *logemes* as the elementary units of cross-cultural logical practice and provided a homotopy-theoretic criterion for their identity. This framework:

- Explains "logical monism" in history (Lemanski 2025) by using different combinations of logemes;
- Mathematically validates claims about "common cores" in diagrammatic reasoning (Anger et al. 2022);
- Explains the structural coincidence of Stoic and Yogācāra inference systems as homotopy invariance;
- Grounds the "logic of diagrams" in pure mathematics, due to univalence.

The earliest attested logemes (*modus ponens* and *modus tollens* in Mesopotamian divination texts) suggest a diffusionist hypothesis: the shared homotopy type of Hellenistic and Buddhist logemes likely reflects historical transmission along the Silk Road, not independent convergence.

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