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**ARTICLE**

# Entanglement and smooth geometry in 4-spacetime

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**Abstract**

We try to understand quantum entanglement by geometric relations of spatially separated regions of 4-spacetime. The relations become detectable by working in the Euclidean smoothness structure underlying the Lorentzian structure. There are 5-dimensional bridges, i.e. 5-dimensional nontrivial smooth $h$-cobordisms connecting spatially separated 4-regions of spacetime. These connections are nonlocal in spacetime. At the quantum regime spacetime is reduced to its smooth atlases of charts which are related by automorphisms of the maksimal Boolean algebra in the quantum lattice of projections. Quantum entanglement in 4-spacetime can be represented by exotic smoothness structures on $\mathbb{R}^4$, which are determined by the $h$-cobordisms due to the results in particular by Casson, Akbulut, Freedman, Donaldson or Gompf. The involutions of corks correspond to the phases between the Boolean ZFC-models and to the change of the exotic $R^4$ in $W^5$. This work is more a description of the ongoing project than a detailed presentation of the results. The discussion focuses on certain general contexts and even philosophical features.

**Keywords:** exotic $R^4$s and 5-cobordisms in spacetime, quantum entanglement, boolean ZFC models

## 1. Introduction

The breakthrough discoveries in differential topology and geometry from 1980s led to the existence of exotic smooth versions of many compact and noncompact 4-manifolds. Since then, researchers have been trying to understand the importance of this phenomenon in physics. A particular interest is the case of exotic $R^4$ and $S^3 \times \mathbb{R}$. Both are open smooth 4-manifolds. Exotic $R^4$s are topologically equivalent (homeomorphic) with $\mathbb{R}^4$ but nondiffeomorphic with standard smooth $\mathbb{R}^4$. Similarly, exotic $S^3 \times \mathbb{R}$ are homeomorphic with $S^3 \times \mathbb{R}$ but nondiffeomorphic with standard smooth $S^3 \times \mathbb{R}$. All known exotic $R^4$s fall into two uncountably infinite classes, small and large exotic $R^4$s. Small are embeddable in standard smooth $\mathbb{R}^4$, while large cannot be embedded in $\mathbb{R}^4$.

Consider a topological 5-cobordism $W^5(N_1, N_2)$, where the boundary of the manifold $W^5$, $\partial W^5$, is the disjoint sum of two topological 4-manifolds $N_1, N_2$, i.e., $\partial W^5 = N_1 \cup N_2$. When $N_1$ is topologically equivalent (homeomorphic) to $N_2$, $N_1 \simeq N_2$, then $W^5 \simeq N_1 \times I$ where $I = [0, 1] \subset \mathbb{R}$ and we call such cobordism $W^5(N_1, N_2)$ *topologically trivial*. To be more precise, we require that the topological manifolds $W^5, N_1, N_2$ be simply-connected and compact and $W^5, N_1 \times I, N_2 \times I$ be homotopically equivalent to each other. Then the statement that they are homeomorhic is the topological $h$-cobordism theorem for $W^n$, $n = 1, 2, ..., k, ...$ and $N_1^{n-1}, N_2^{n-1}$ ($h$ stays for the above mentioned homotopical equivalence). The proof for the $W^5$ case was given in Freedman 1982.

We want to model nonlocal connections between regions of spacetime manifold by 5-cobordisms, such that this can shed light on the phenomenon of quantum entanglement. However, certain clarifications are in order. Spacetime is a smooth Lorentzian 4-manifold, but its underlying manifold (mathematically, which can be non-physical) is an Euclidean open smooth 4-manifold on which the Lorentzian structure is introduced. Thus, open regions $U_1, U_2$ in spacetime should be subsets of $N_1, N_2$ correspondingly, and all manifolds $W^5, N_1, N_2, U_1, U_2$ should be smooth. Then we deal with smooth $h$-cobordism 'theorem' rather than the topological one. However, in dimension $n = 5$ the statement:

*Given a smooth h-cobordism $W^5$ between smooth, compact, Euclidean, simply-connected 4-manifolds $N_1, N_2$ such that $W^5, N_1 \times I, N_2 \times I$ all are pairwise homeomorphic, the smooth cobordism $W^5(N_1, N_2)$ is diffeomorphic to $N_1 \times I$*

is, in general, false. This was shown in Donaldson 1983. The understanding of this phenomenon is deeply rooted in modern differential topology and geometry and is crucial for this work.

The quantum uncertainty principle between some pairs of observables and quantum entanglement between two systems are nontrivially related. Entanglement can be detected as non-classical (e.g. breaking the Bell's bound) if there are noncommuting observables $[X, Y] \neq 0$ at Alice and Bob sides. Certainly, entanglement cannot erase the uncertainty of the observables applied to each system separately, but it can reduce the quantum bound for the entangled state when correlations are between the results of Alice $a(X), a(Y)$ on the entangled state and the results of Bob $b(X), b(Y)$ on it. An entangled state is just a state and it does not require any noncommuting observables (or uncertainty between them). But to witness entanglement as a quantum phenomenon we need noncommuting $X, Y$. Similarly, entanglement can be detected in spacetime by measuring the momentum and position $P, Q$. For a single system, the entanglement of states cannot reduce the Heisenberg uncertainty, however, for two entangled subsystems $s_1 \otimes s_2$ the $P, Q$ measurement in the Alice and Bob laboratories can reduce the uncertainty, which, though, does not contradict the uncertainty relations in each subsystem individually (Horodecki et al. 2009).

Here we try to understand the nonlocality in spacetime by incorporating 5-dimensional bridges corresponding to nontrivial $W^5$ cobordisms. These bridges introduce additional correlations between spatially separated noncommuting observables.

## 2. 5-dimensional bridges in 4-spacetime and exotic smoothness

This and the following sections collect arguments for seeing nontrivial smooth cobordisms $W^5$ as modeling the spacetime bridges that connect 4-regions $U_1, U_2$ . Here we focus on the classical differential geometric point of view, while in the next section we connect this with the formalism of quantum mechanics (QM). As we mentioned above, we work in Euclidean metrics, since they underlie the Lorentzian case, and the smoothness structures assigned to Euclidean $\mathbb{R}^4$ allow for uncovering a fundamental mathematical layer that has connections to QM. We use the symbol $R^4$ for exotic $\mathbb{R}^4$s where $\mathbb{R}^4$ refers to the standard smoothness structure on $\mathbb{R}^4$. Let us follow the idea in Król and Asselmeyer-Maluga, 2025 that under certain, rather extreme conditions in spacetime, one can separate $p = (p_1, p_2, p_3, E)$ and $q = (q_1, q_2, q_3, t)$ as coordinates of a single physical object, and they contribute as local coordinate frames in the reconstructed defragmented new spacetime. Here we describe the separation process as the evolution of these 4-dimensional patches $p, q \simeq \mathbb{R}^4$ that are represented by trivial open 5-cobordisms $C_0 = \mathbb{R}^4 \times I$ as the subcobordism of certain trivial compact $W^5(N_1, N_2) \simeq N_1 \times I$, thus, $C_0 \subset W^5$. So, the evolution of $p$ or $q$ can be represented by trivial cobordisms as in Fig. 1a) when the momenta and positions to be measured are actually assigned to different particles. However, in the case where momentum and positions are observables of a single particle, the uncertainty relation constrains their simultaneous measurement

$$\Delta\overline{p} \cdot \Delta\overline{q} \geq \frac{\hbar}{2}.$$

This 3-dimensional version can be formally extended over dimension four as follows

$$p^{\mu} \cdot q_{\nu} \geq \frac{\hbar}{2}\delta^{\mu}_{\nu}, \mu, \nu \in \{0, 1, 2, 3\}, \; p_{\mu} = (E, \overline{p}), q_{\nu} = (t, \overline{x}).$$

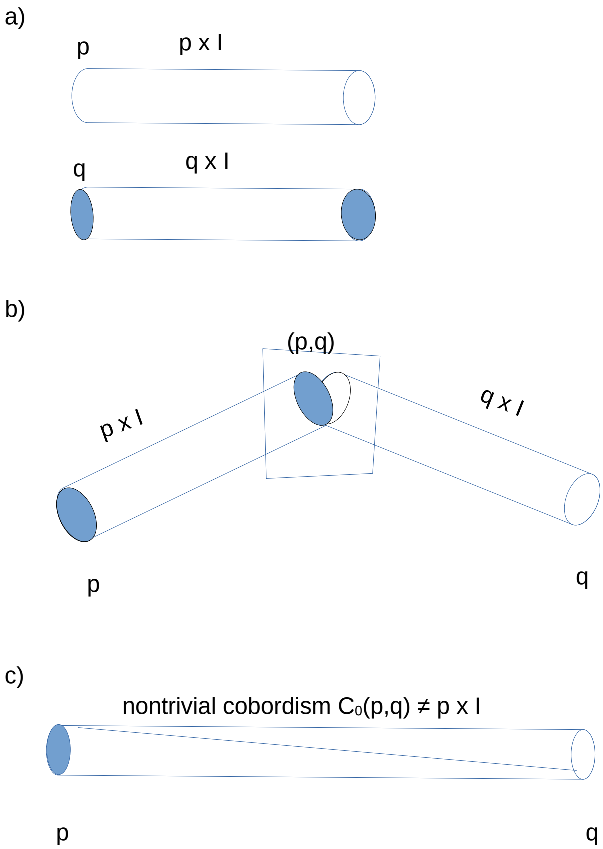


**Figure 1.** 5-dimensional evolutions of 4-regions. a) $p, q$ evolve via trivial cobordisms $p \times I, q \times I$. This corresponds to the independent shifts of 4-dimensional local regions in spacetime as can be the case for two separated particles such that their positions and momenta are not confined by the uncertainty reltion. b) Uncertainty relations in a small 4-region $(p, q)$ affect the evolutions of the regions $p$ and $q$ and enforces the correlations. c) There results a nontrivial 5-cobordism $C_0(p, q)$ as a subcobordism of certain $W^5(N_1, N_2)$.

These 4-dimensional coordinate patches $\mathbb{R}^4 \simeq p, q$ can evolve in 5-dimensions as 5-cobordisms, leading to their spatial separation with a degree of uncertainty being transferred to entanglement. The quantum side of the process will be discussed in the next section. Here we want to determine the smooth geometric component of it. The evolution of the $p, q$ patches that undergo the uncertainty principle in the micro initial state is represented in Fig. 1b). In Król and Asselmeyer-Maluga, 2025 conditions were given for the destruction of the integrity and causality of the spacetime manifold and this resulted in the fragments of spacetime, each being a $\mathbb{R}^4$ local patch. Then, the reverse process was described to retain the integrity, causality, and the underlying smoothness structure. The smooth regions of the reconstructed Euclidean spacetime become necessarily exotic smooth $\mathbb{R}^4$, i.e. $R^4$. Here we focus on this classical geometric picture (without touching quantum level) and describe the appearance of $R^4$s in spacetime due to nontrivial cobordisms $W^5$ connecting 4-

regions $p, q$ in spacetime. This is represented in Fig. 1c) where nontrivial open 5-subcobordism $C_0(p, q) \subset W^5(N_1, N_2)$ is visualized. The classical counterpart to losing the causality and integrity of 4-spacetime is to consider the 5-dimensional $C_0$ that cannot sit in $M^4$ and thus cannot be local in spacetime. We will see that indeed $C_0$ contains information about quantum entanglement.

Why $C_0$ has to deal with exotic $R^4$s? This is really a deep result in mathematics first observed and proved in Donaldson 1983 by applying gauge field theory methods to the differential geometry of 4-manifolds and making use of another profound result in the topology of 4-manifolds obtained in Freedman 1982. The analysis of the nontrivial cobordism $W^5(N_1, N_2)$ for compact 4-manifolds $N_1, N_2$ that were simply connected such that $N_1 = K3\#\overline{CP^2}$ and $N_2 = \#CP^2\#20\overline{CP^2}$ led Akbulut to the discovery of compact cork (Akbulut cork) $K \subset N_1$ with a nonempty boundary $\partial K$. Here $K3$ is the celebrated $K3$ - surface (complex 2-dimensional compact smooth manifold), $CP^2$ is the 2-dimensional complex projective space, $\overline{CP^2}$ is $CP^2$ with the reversed orientation and # is the connected sum of two manifolds (obtained by cutting off 4-balls from each manifold and gluing the remnants manifolds by homeomorphism, or difffeomorphism, of their common boundary). It appears that in the above case the Akbulut cork $K$ has an explicit description via handle decomposition, and this is known as Mazur manifold (Akbulut, 1991(a), 1991(b)) and $\partial K = \Sigma(2, 5, 7)$ is the homology 3-sphere, one of Brieskorn's spheres (Gompf and Stipsicz 1999). The first contractible 4-manifolds whose nonempty boundary would not be $S^3$ were constructed by Mazur in 1961. Then many other examples were built and it was shown that the boundary of such generalized Mazur manifolds can be Brieskorn homology 3-spheres $\Sigma(2, 5, 7), \Sigma(3, 4, 5), \Sigma(2, 3, 13)$ (Akbulut and Kirby, 1979). Since then it has been shown (in particular by Casson) that many other Brieskorn spheres appear as the boundary of such contractible 4-manifolds. Thus, the point is that for any nontrivial smooth 5-cobordism $W^5(N_1, N_2)$ between two smooth nondiffeomorphic, compact, simply-connected 4-manifolds $N_1, N_2$, there is always an embedded cork $K \subset N_1$ and the special role is played by $\partial K$. The celebrated example is $N_1 = K3\#\overline{CP^2}$ and $N_2 = \#CP^2\#20\overline{CP^2}$, where these manifolds are homeomorphic but nondiffeomorphic, and the Akbulut cork $K$, which is a compact, contractible 4-manifold, has a boundary $\partial K = \Sigma(2, 5, 7)$.

The following decomposition result is basic for understanding 5-$h$-cobordisms and their relation to exotic $R^4$s (Curtis et al., 1996, p.343). Let $N_1, N_2$ be two smooth compact simply-connected 4-manifolds and $W^5(N_1, N_2)$ be a smooth nontrivial simply-connected 5-cobordism between them.

**Theorem 1** *There exist decompositions: $N_1 = K \cup_{\partial K} N_1'$ and $N_2 = K \cup_{\partial K} N_2'$ such that $(N_1', \partial K)$ and $(N_2', \partial K)$ are diffeomorphic and the subcobordism $W^{5\prime}(N_1', N_2') = N_1' \times I$ is trivial.*

$K$ is the compact contractible 4-manifold (a cork) with a nonempty boundary $\partial K$. In conclusion, we have

**Theorem 2** *a) There exist two nondiffeomorphic exotic $\mathbb{R}^4$s, $R_1^4$ and $R_2^4$ such that $K \subset R_1^4 \subset N_1$ and $K' \subset R_2^4 \subset N_2$. b) $K'$ is diffeomorphic to $K$ relative to the nontrivial involution of its boundary $\tau(\partial K)$.*

The decomposition of an $h$-cobordism $W^5(N_1, N_2)$ is shown schematically in Fig. 2

A. Casson also showed that the nontriviality of 5-cobordism can be obtained by cutting off $R_1^4$ in $N_1$ and gluing it back by involution of its end in infinity. In fact, the reduction of this Casson 'non-compact' procedure to compact $K$ and the involution of $\partial K$ was quite a surprise. Finally, it was shown in Gompf, 2018 that these two procedures can be used interchangeably and the subsequent generalization of this is possible to the $G$-slice corks and $G$-slice exotic $R^4$ for quite general groups $G$.

For the canonical example recognized by Akbulut we have $\partial K = \Sigma(2, 5, 7)$ and its involutions give rise to the change of the smoothness structures from $R_1^4$ to $R_2^4$. Note that such a change of the smoothness structure on $\mathbb{R}^4$ cannot be described within GR formalsim, as well as the 5-cobordisms connecting regions of 4-dimensional spacetime are not entities describable on $M^4$. This is the motivation for considering some extension of GR which would lead to a close relation with the QM formalsim.

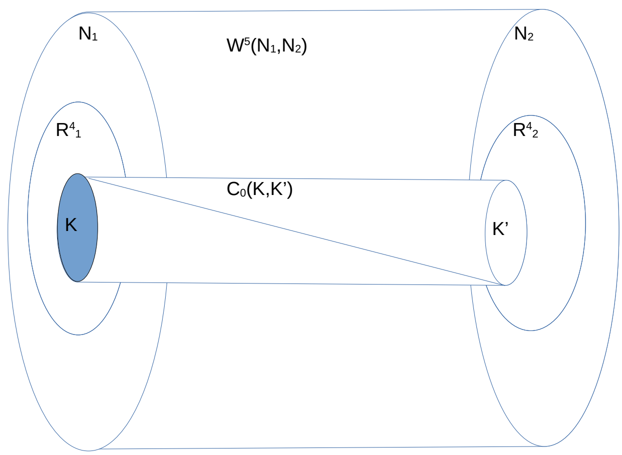


**Figure 2.** The decomposition of a nontrivial $h$-cobordism $W^5(N_1, N_2)$.

## 3. QM and the evolution of exotic spacetime 4-regions

In this section, we present arguments that the QM formalism can support the existence of exotic 4-regions in spacetime and their evolution as 5-cobordisms. The exotic $R^4$s here are small, i.e., embeddable in standard $\mathbb{R}^4$. The 5-cobordisms certainly cannot be embedded in 4-spacetime and can be a source of certain nonlocality detected in spacetime. However, some additional conditions must be met. We start with the local modeling of spacetime by $R^4$s patches that belong to local Boolean models $V^B$ rather than to any universal a priori established universe of sets like the Von Neumann cumulative proper class model $V$ (reference to the constant $V$ is a usual practice in physics.). The varying local ZFC models do not follow any sheaf category localizations but rather are hybrid methods of Boolean models and the relations between them. Reference to the local patches of the spacetime 4-manifold (before the Lorentzian twist) as embedded also in local Boolean-valued models $V^B$ has a nice advantage when one thinks about unification of GR and QM. This is the extension of the diffeomorphism group of $M^4$ by the group of morphisms between $V^B$s where the latter are generated by $Aut\, B$.

$$Diff(\mathbb{R}^4) \oplus Aut\, V^B.$$

Thus, the equivalence principle of GR gains additional degrees of freedom connected with the automorphisms of $V^B$ such that the local choice of flat $\mathbb{R}^4$ where gravitational energy vanishes, is augmented by the local phase connected with the automorphism of models where $\mathbb{R}^4$ patches are living, $R^4_{V^{B1}} \in V^{B1}$. Two flat local coordinate patches $\mathbb{R}^4_1, \mathbb{R}^4_2$ in addition to their flatness can be distinguished by their local phase:

$$R^4_{V^{B1}} \simeq_F R^4_{V^{B2}} \text{ where } F : V^{B1} \to V^{B2} \text{ is the relative phase.}$$

Instead of $R^4_{V^{B1}}$ we can write $R^4_1$ and this is the product $R \times R \times R \times R$ in the ZFC model $V^{B1}$ and this $R^4_1$ has nothing to do with exotic $R^4_1$ as before. The context will clearly distinguish both objects. Now the point is that this phase $F$ has QM origins and on infinite-dimensional Hilbert spaces $\mathcal{H}^\infty$ can be further reduced to the automorphisms of a Boolean algebra $B$. So, the extended equivalence principle taking into account the relative phase $F$ would read as follows.

> [ExtEP] There is always possible in the microscale to choose a local coordinate frame $U = \mathbb{R}^4$ and a Boolean ZFC model $V^B$ that it would erase the gravitational effects in $U$ and $U$ would be entangled with another possibly spatially separated 4-region $U' \simeq U$ such that their entanglement induces gravitational effects nonlocalized to $U$.

We want to show that this ExtEP is probable by giving step by step explanation, but a more rigorous justification will be published elsewhere. Let us reformulate ExtEP as

> [ExtEP'] The choice of the local $\mathbb{R}^4$ patch in microscale in spacetime that erases all gravitational effects always factorizes through exotic $R^4$ and leads to the quantum entanglement effects.

A full justification of this very strong statement will be the subject of a separate publication. Here we focus on some elements of ExtEP' making it probable. Let $U = \mathbb{R}^4$ and $\dim \mathcal{H} = \infty$. The ZFC twist of QM (Król and Asselmeyer-Maluga, 2025) determines $F : V^{B1} \to V^{B2}, F \not\simeq Id$ and:

a) a Boolean ZFC model $V^B$ where $U = R^4_B$;
b) a Boolean ZFC model $V^{F(B)}$ where there is an accompanied flat $U_F = R^4_{F(B)}$;
c) exotic $R^4_1$ where $U_B$ is a flat local coordinate patch;
d) exotic $R^4_2$ where $U_{F(B)}$ is a local flat coordinate patch.

a) means that once local flat $U' = \mathbb{R}^4$ is obtained in spacetime erasing gravitational effects, which is legitimate due to EP, at the micro level there is room for a deeper representation of $U' = \mathbb{R}^4$ as $U = R^4_B$ in certain ZFC Boolean model $V^B$. Even though the models $V^B$s vary depending on the specific region of spacetime, for $\dim \mathcal{H} = \infty$ there exists just a single model $V^B$ where $B$ is the atomless Boolean measure algebra. The different regions in spacetime refer to different automorphic copies of $F'(V^B)$. However, $F'(V^B)$ are determined by the automorphisms $F$ of $B$

$$\forall_{F'} \exists_F F'(V^B) = V^{F(B)}, F \in Aut\, B.$$

Thus b) holds true.

The justification of c) and d) is based on the relation of the quantum lattice of projections $\mathbb{L}$ for $\mathcal{H}^\infty$ and the spacetime manifold $M^4$ with a smooth atlas $\{U_\alpha\}_{\alpha \in I}$. Again, crucial is the ZFC twist of QM that assigns models $V^{B_\alpha}$ to the local patches $U_\alpha$ such that

$$U_\alpha \simeq R^4_\alpha \text{ in } V^{B_\alpha} \text{ and } V^{B_\beta} = V^{F_{\alpha\beta}(B_\alpha)} \text{ where } F_{\alpha\beta} \in Aut\, B.$$

We can characterize the border line between quantum and classical on the basis of a smooth atlas $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ subordinated to the lattice $\mathbb{L}(\mathcal{H}^\infty)$. Let $B$ be a maximal atomless Boolean subalgebra of projections chosen from $\mathbb{L}$. $B$ is the measure algebra. We call $\mathcal{U}$ a quantum atlas of $\mathbb{R}^4$ if the corresponding projections from $\mathbb{L}$ belong to at least two different maximal algebras.

**Lemma 1** *If* $|\mathcal{U}| \geq 2$ *then* $\mathcal{U}$ *is quantum.*

This follows from the 1 : 1 correspondence between $U_\alpha$ from $\mathcal{U}$ and $V^{B_\alpha}$ which gives at least two maximal Boolen algebras $B \subset \mathbb{L}$ and consequently, it has to be determined at least a pair of noncommuting observables subordinated to the different algebras correspondingly. In addition, we have

**Lemma 2** *(Król et al., 2017, Corr. 1, Th. 3)* If any smooth $\mathcal{U}$ on some $\mathbb{R}^4$ is quantum, then such a smooth $(\mathbb{R}^4, \mathcal{U})$ has to be exotic $R^4$.

On the contrary, $|\mathcal{U}| = 1$ corresponds to the classical case and the standard smooth $\mathbb{R}^4$. Thus we arrive at the conclusion that a quantum system in spacetime can modify its local standard smoothness toward exotic $\mathbb{R}^4$ and this exotic $R^4$ encodes information about the quantum system. The fundamentals are the following

**Lemma 3** *A nontrivial 5-cobordism* $W^5$ *can encode the change of base in Hilbert space* $\mathcal{H}^\infty$.

In particular, the projectors in one base are sent to some projectors in another base.

**Theorem 3** *The quantum entanglement of two spatially separated quantum systems at microscale is represented in 4-spacetime by the relation of two exotic smooth local 4-patches linked by a nontrivial 5-cobordism* $W^5$ *outside of spacetime.*

Instead of the proof, we collect the main points with short commentaries leading to Theorem 3.

P1. Entanglement as a quantum effect in spacetime requires multiple local patches at the initial interaction stage and in the final spatially separated stage.
P2. Any quantum system in spacetime determines a collection of local coordinate frames $\{qU_p\}_{p\in M^4}$ in a way that: Any observable $\mathcal{O}_i, i \in O$ when measured in a state $\psi_0 \in \mathcal{H}^\infty$ determines the value $o_i(\psi_0)$ and the local flat $\mathbb{R}^4_{o_i}$.
P3. On the microscale $\mathbb{R}^4_{o_i}$ becomes $R^4_i$ in the ZFC Boolean model $V^{B_i}$.
P4. When fragmented, the spacetime manifold $M^4$ becomes a collection of flat local $R^4_i, i \in I$ and a set of relations $F'_{ij} : V^{B_i} \to V^{B_j}$ between ZFC models hosting $R^4_i$s.
P5. If the observables $\mathcal{O}_i, \mathcal{O}_j, i \neq j$ are compatible, i.e., measured simultaneously, then $V^{B_i} = V^{B_j}, F'_{ij} = id$. If $[\mathcal{O}_i, \mathcal{O}_j] \neq 0$ then $F'_{ij} \neq id$ and $R^4_{o_i}$ is not identically diffeomorphic to $R^4_{o_j}$.

Fragmentation in P4. was explained in detail in Król and Asselmeyer-Maluga, 2025. The microscale appearance of ZFC Boolean models in spacetime regions (P3.) has been systematically explored in Król and Asselmeyer-Maluga, 2020; Król, Bielas, and Asselmeyer-Maluga, 2023. P2. and P5. are direct consequences of the construction, and P1. follows from them and Lemma 1.

## 4. Discussion and perspectives

Theorem 3 opens several opportunities related to entanglement and its relation to spacetime regions; however, they need a more thorough explanation. Nonlocality in this context means a relation of two 4-regions that is nonlocal in spacetime. Note that the bridge $W^5(N_1, N_2)$ connecting the regions is 5-dimensional and cannot be local 4-dimensional. However, these geometric data of $W^5$

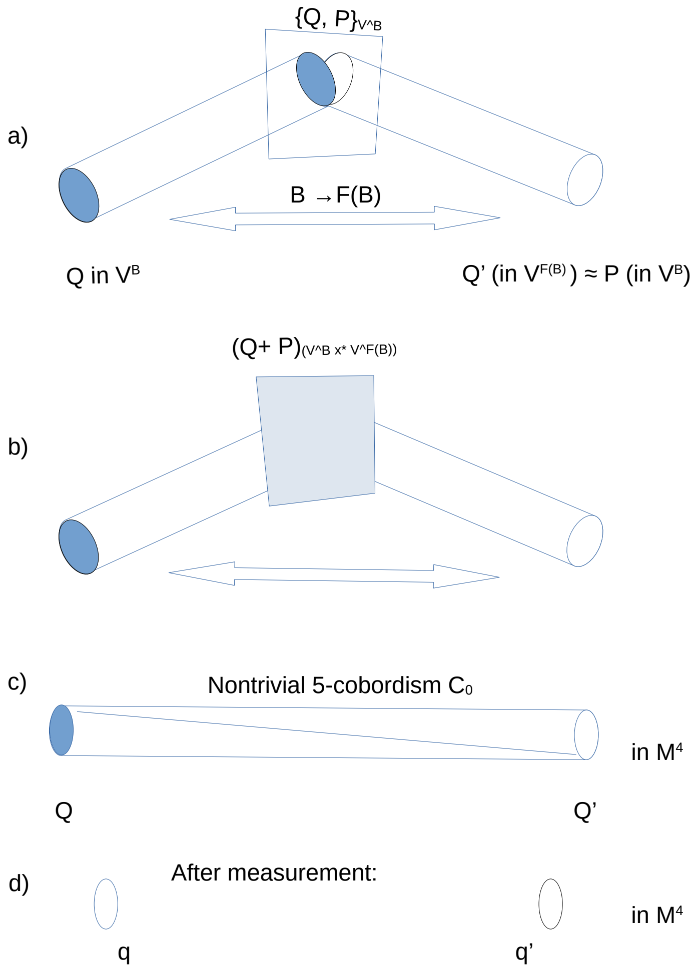


**Figure 3.** Entanglement and uncertainty in spacetime. a) In some conditions $q$-coordinates and $p$-coordinates assigned to a single particle can become separated in 4-dimensions. Here $p, q$ refer to $(\overline{p}, E)$, $(\overline{x}, t)$ correspondingly. They become 4-dimensional local $\mathbb{R}^4$ patches, however with the relative phase $F \in Aut(B)$. b) $Q, P$ become entangled before measurement in 4-spacetime $M^4$. This leads to $(P + Q)$ in the composed ZFC model $V^B x * V^{F(B)}$. c) The spacetime separation of $P, Q$ and the entanglement of them leads to the entangled regions in $M^4$. The measurement of a particle's $P$ in $V^B$ is entangled with the measurement of $Q'$ in $V^{F(B)}$ since $Q'$ in $V^{F(B)}$ corresponds to $P'$ in $V^B$ and the momentum is preserved in $V^B$. d) After measurement the 4-regions of $P$ and $Q$ coordinates contribute to spacetime as independent local regions.

should be augmented with a suitable quantum content to represent the entanglement. We will see that it is possible and that $W^5$ can represent a quantum entanglement.

First, following Lemma 1 if any open atlas $\mathcal{U}$ of $\mathbb{R}^4$ fulfils $|\mathcal{U}| \geq 2$ then it can encode quantumness. This relies on the direct observation that if there is $1:1$ correspondence of the maximal Boolean algebras in $\mathbb{L}$ with open maps in $\mathcal{U}$ then given at least two distinct such maps we have to have at least a pair of noncommuting observables $\mathcal{O}_1, \mathcal{O}_2, [\mathcal{O}_1, \mathcal{O}_2] \neq 0$ or an observable $\mathcal{O}$ with at least two nonperpendicular but different positive operator-valued measure (POVM) eigenspaces. However, to measure entanglement in spacetime we need noncommuting observables let them be $X, Y, [X, Y] \neq 0$. Otherwise, the Bell-like inequality (e.g. CHSH) will never be broken, or the classical bound would never be exceeded. That is why to see entanglement as witnessed by $X, Y$ in spacetime one needs $[X, Y] \neq 0$ and $X, Y$ to be applied to both entangled systems. This is the reason for $|\mathcal{U}| \geq 2$ as above.

It follows another strong consequence, as stated in Lemma 2: *Any smooth quantum* $\mathbb{R}^4$ *has to be exotic* $R^4$. Thus, we find another heuristic justification for Theorem 3.

Let us derive a direct description of the nonlocal entanglement of observables $[X, Y] \neq 0$ in terms of $W^5$. Let $a(X), a(Y), b(X), b(Y) \in \{a_1, a_2, b_1, b_2\}$ be the possible outcomes of Alice and Bob. Let the phase $F' : V^{B1} \to V^{B2}$ sends $X$ to $Y$. Thus, measuring $X$ on Alice side gives rise to measuring $Y$ on Bob's side. There are sets of local patches $U_1, U_2, U_1', U_2'$ for Alice and Bob (primes), respectively. To clarify presentation, let us refine the correspondence (which does not affect the construction) *opens in* $M^4$ *to operators*: 'subfamily $\{U_i, i \in J\}$ of open local patches of $M^4$' $\leftrightarrow$ 'eigenvalues $\{o_i \in \mathbb{R}, i \in J\}$ of any observable $\mathcal{O}$'. This subfamily has relative phase $F' : V^B \to V^B$ identity on $V^B$, since the projections on the eigenspaces of the same $\mathcal{O}$ commute. Still, the relative diffeomorphisms are not identities $f : \mathbb{R}^4 \to \mathbb{R}^4, f \neq id$ and this is the refinement that extends. The minimal case states that there are at least two eigenvalues and two corresponding open charts $(o_1, o_2; U_1, U_2)$ all in the same model $V^{B1}$. So, thus, $U_1, U_2$ are opens of $R_1^4 \in V^{B1}$ and any open cover of $R_1^4$ has a cover that does not allow for a single local chart $R_1^4$. This also follows from taking two noncommuting $X, Y$ (which is essential to detect entanglement) and that $X, Y$ are in the same model $V^{B1}$ (Alice) and $V^{B2}$ (Bob). This means $R_1^4$ is already exotic in $V^{B1}$ (Lemma 2). A smooth open nontrivial subcobordism of $W^5$ sends $R_1^4$ to $R_2^4$ in $V^{B2}$ with the nontrivial relative phase $F' : V^{B1} \to V^{B2}, F' \neq Id$. Then $R_2^4$ is exotic in $V^{B2}$. The minimal extension as above on the commuting eigenprojections of $\mathcal{O}$ assigns the local charts $U_1, U_2$ of $M^4$ (for any smooth atlas) with nonidentity diffeomorphisms as transition maps. After measurement, the result is described in one frame, $U_1$ corresponding to $a_1$ or $U_2$ for $a_2$ in $V^{B1}$. Similarly for the image $U_1'(b_1)$ or $U_2'(b_2)$ in $V^{B2}$. Let us assume that $U_1, U_2$ in exotic $R_1^4$ correspond to the ends $E_1, E_2$ of $R_1^4$ (Gompf and Stipsicz 1999). Then, in $R_2^4$ they are assigned to the ends with the following twist

$$U_1 \to E_1 \to U_1' \to E_2' \text{ in exotic } R_2^4 \text{ in } V^{B2}$$
$$U_2 \to E_2 \to U_2' \to E_1' \text{ in exotic } R_2^4 \text{ in } V^{B2}.$$

The entanglement of 4-regions in spacetime causes the instantaneous action at a distance: *a measurement of* $a_1$ *in* $U_1$ *in* $R_1^4$ *enforces the measurement of* $b_2$ *in* $U_2'$ *in* $R_2^4$. The subtlety is a hidden, though general property of spacetime: in the quantum regime open covers replace the spacetime manifolds. In other words, *spacetime is defined categorically in the classical regime from open covers.* Thus, quantum measurement in spacetime determines local micro patches which are related by the 5-dimensional bridges-cobordisms.

The entire process behind the twist of local patches at spatially separated regions is based on Theorem 2 and its equivalence to the twist of the ends at infinity of exotic $R_1^4$ to obtain $R_2^4$. This twist is generated by the nontrivial phase $F_{12}' : V^B \to V^B$ (Figs. 2, 3).

The appearance of such 5-dimensional bridges bears certain similarity to the recently formulated proposal that possibly entanglement is nonlocal in spacetime due to connections via wormholes. Currently we do not have arguments allowing for association of our geometric 5-bridges with Suskind's proposal. Let us note only that the source of exotic smoothness in spacetime has been assigned to fragmentation due to extreme conditions in the singular regions like in black holes (Król and Asselmeyer-Maluga, 2025). Whether similar conditions can generate an exotic matter needed for opening wormholes has not been specified so far.

The appearance of exotic $R^4$s in the quantum regime of spacetime is quite important and requires discussion. Let us turn our attention to topological quantum field theory (TQFT) where the relation of $n$-cobordisms and Hilbert spaces is in the heart of the constructions. The original formulation in Atiyah, 1988 defines a TQFT symmetric monoidal functor $Z : \mathbf{Cob}_n \to \mathbf{Hilb}$ from the category

**Cob**$_n$ of $n$-cobordisms (morphisms) between $n-1$ compact manifolds (objects) to the category of finite-dimensional Hilbert spaces (objects) with linear operators (morphisms). This axiomatic version of TQFT does not allow infinite-dimensional $\mathcal{H}^\infty$. The reason is dualizability in **Hilb**, i.e. the existence of evaluation $\epsilon$ and coevaluation $\eta$

$$\epsilon : \mathcal{H}^* \otimes \mathcal{H} \to \mathbb{C}; \quad \eta : \mathbb{C} \to \mathcal{H}^* \otimes \mathcal{H}.$$

To be well-defined, theese operations require finite-dimensional $\mathcal{H}$ and they allow for essential for Atiyah - Segal TQFT gluing behavior

$$Z(M \cup_N M') = Z(M) \circ Z(N^{-1}) \circ Z(M'). \tag{1}$$

So, dualizability hence finite-dimensional $\mathcal{H}$ is essential. In the context of physics, $\epsilon$ corresponds to anihilation, while $\eta$ corresponds to the birth of states or particles. In our case of nontrivial 5-cobordisms, local representability of atlases in $V^B$ requires $\dim \mathcal{H} = \infty$ since then $B = Bor\mathbb{R}/Null$ is the measure algebra that is universal among all maximal projection algebras in $\mathbb{L}(\mathcal{H}^\infty)$. We claim that this is not an accident, but rather a very crucial property of the formalism.

First, the extension of TQFT over $\mathcal{H}^\infty$ would require complete extensions of the formalism, i.e. the use of higher categories up to $\infty$-categories (Lurie 2009). Then one has to replace the target category of finite-dimensional Hilbert spaces $(Hilb, \otimes)$ by a symmetric monoidal category in which the objects assigned to 4-manifolds are dualizable in a way to support the gluing property (1). This would be a way toward grasping the nontrivial smooth 5-cobordisms in the TQFT formalism and eventually understanding the nonlocality of entanglement between the regions of spacetime. In fact, the realization of the above would require quitting the TQFT structure and making smooth variants of rigourous categorical field theories where various modifications of the tangential categorical structure would be required.

Second, the inclusion of exotic smoothness on $\mathbb{R}^4$ in the TQFT formalism is again (if possible at all) based on the extension of TQFT over dualizability realized in higher categories, as mentioned above (Grady and Pavlov 2021). In particular, the extended TQFT should refer to infinite dimensional $\mathcal{H}$. This kind of cobordisms between open 4-manifolds would become the main player of the approach.

Our approach strongly indicates the role of $\mathcal{H}^\infty$, exotic $R^4$s and nontrivial 5-cobordisms. However, this comes from completely different points of view based on the automorphisms of $B$. As we presented before, the $\infty$ dimension of $\mathcal{H}$ gives rise to the universal $B = Bor(\mathbb{R})/Null$ for entire $\mathbb{L}(\mathcal{H}^\infty)$ but there is also the level of $Aut\,B$ that distinguishes cases of finite and infinite dimensions.

**Lemma 4** *If a maximal complete Boolean algebra of projections in $\mathbb{L}$ is atomic, then all automorphisms of $B$ are extendable to the global automorphisms of $\mathbb{L}$.*

**Lemma 5** *If a maximal complete Boolean algebra of projections is atomless then there exist automorphisms of $B$ that are nonextandable over $\mathbb{L}$.*

But $B = Bor(\mathbb{R})/Null$ is atomless, and thus there are local nonextendable automorphisms of $B$ for $\mathcal{H}^\infty$. Moreover, these local automorphisms apply as local phases (gauges) in the description of exotic $R^4$s. Schematically, the local phase represented in Fig. 3a as $F$ where $Aut\,B$ generates $Aut(V^B)$ and gives rise to the change of the exotic structure on $R^4_1$. The nontrivial 5 subcobordism of two open exotic $R^4$s emerges (see Fig. 2) which is generated by the inversion of the boundary of the Akbulut cork $\Sigma(2,5,7)$. This inversion $\tau : \Sigma(2,5,7) \to \Sigma(2,5,7)$ is due to the nontrivial local phase between $R^4_1 \to R^4_2$. Thus finally, the extension over infinite dimensional Hilbert spaces and allowing for 5-cobordisms between open 4-manifolds result from the bottom up approach reviewed in this work and lies in the heart of top down approaches in higher categories and TQFT. We believe that this phenomenon is the key toward better understanding the discripancy of QM and GR.

Let us indicate yet another feature connected to the above. The negation of the Tsirelson conjecture says roughly that the infinite dimension of $\mathcal{H}$ allows us to distinguish by finitely many quantum correlation the situation where on $\mathcal{H}^\infty$ two parties' measurements of commuting observables and the case where we have factorized Hilbert space $\mathcal{H}_A^{\infty,\text{finite}} \otimes \mathcal{H}_B^{\text{finite},\infty}$ and the measurements are on the factors independently. It appears that the nonfactorizable case is more strongly correlated than the factorizable case. The original prove goes through Touring uncomputability of certain sets of formulas describing both cases and showing that they differ by the degree in Touring classes. Recently, an alternative proof was given by methods of maximal Boolean algebras $Bor(\mathbb{R})/Null$ in the lattice $\mathbb{L}$ and forcing relation in models of ZFC (Król and Asselmeyer-Maluga 2024). This indicates that the coding of infinite dimension of $\mathcal{H}^\infty$ along with the ZFC twist of QM are well-suited for analyzing subtle phenomena of randomness in QM.

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