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ARTICLE

Quantum Weak Values and Harmonic Analysis on Lie Groups

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Abstract

The aim of this contribution is to generalize a formula proved by Maurice de Gosson (de Gosson 2017) about weak values in the context of the phase-space formulation of Quantum Mechanics (Rundle and Everitt 2021), in order to express those weak values using tools coming from the harmonic analysis on Lie Groups (Faraut 2006). A general formula which enables us to compute weak values is proved, in which the integration on a Lie Group is substituted to the integration on phase-space, using Haar measures. Then this formula is applied to SU(2) and SO(3) and also to the quotient group G/H, where H is a normal subgroup of G.

Keywords: Grossman-Royer, Weyl-Heisenberg, weak values, Lie groups, pre- and post-selection, Haar measure, group representation, special unitary group SU(2), special orthogonal group SO(3).

1. Introduction

The aim of this contribution is to generalize a formula proved by Maurice de Gosson (de Gosson 2017) about weak values in the context of the phase-space formulation of Quantum Mechanics (Rundle and Everitt 2021), in order to express those weak values using tools coming from the harmonic analysis on Lie Groups (Faraut 2006)).

Maurice de Gosson (de Gosson 2017, pp.151-153) has shown that we can express a weak value in general using the Cross-Wigner transform:

$$\langle \hat{A} \rangle_{\psi_f \psi_i} := \frac{\langle \psi_f | \hat{A} \psi_i \rangle}{\langle \psi_f | \psi_i \rangle} = \int_{\mathbb{R}^{2n}} dz \, \frac{W(\psi_f, \psi_i)(z)}{\langle \psi_f | \psi_i \rangle} \, a(z) \tag{1}$$

where $a(z)$ is the *Weyl symbol* of the operator $\hat{A}$ and $W(\psi_f, \psi_i)$ the *Cross-Wigner Transform* given by:

$$W(\psi_f, \psi_i) = (\frac{1}{2\pi\hbar})^n \int_{\mathbb{R}^n} dy \, e^{-i\frac{py}{\hbar}} \, \overline{\psi_f(x - \frac{y}{2})} \, \psi_i(x + \frac{y}{2}) \tag{2}$$

with $z = (x, p)$ a point of the phase-space $\mathbb{R}^{2n}$; x, y and p are vectors.

In the particular case where $\hat{A}$ is, for example, the projector $\Pi_{\psi_r} = \frac{|\psi_r\rangle\langle\psi_r|}{\langle\psi_r|\psi_r\rangle}$

$$\langle \Pi_{\psi_r} \rangle_{\psi_f \psi_i} := \frac{\langle \psi_f | \Pi_{\psi_r} \psi_i \rangle}{\langle \psi_f | \psi_i \rangle} = (2\pi\hbar)^n \int_{\mathbb{R}^{2n}} dz \, \frac{W(\psi_i \psi_f)(z)}{\langle \psi_f | \psi_i \rangle} \, W(\psi_r)(z) \tag{3}$$

The Cross-Wigner transform is related to the Weyl-Heisenberg groups acting on phase-space. This is clear if we express $\langle\hat{A}\rangle_{\psi_f\psi_i}$ as follows, according to de Gosson (de Gosson 2017), pp. 17 (2.1), 151 (12.18)), successively:

$$W(\psi_f, \psi_i) = (\frac{1}{\pi\hbar})^n \, (\hat{R}(z)\psi|\phi)_{L^2}$$

$$\langle\hat{A}\rangle_{\psi_f\psi_i} = \frac{1}{(\psi|\phi)} \int_{\mathbb{R}^{2n}} dz \; a(z) \; W(\psi, \phi)$$

$$\langle\hat{A}\rangle_{\psi_f\psi_i} = \frac{1}{(\pi\hbar)^n} \int_{\mathbb{R}^{2n}} dz \; a(z) \; \frac{\langle\hat{R}(z)\psi_f|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} \tag{4}$$

The operator $\hat{R}$ is the so-called Grossman-Royer operator which is nothing but the Stratonovich-Weyl kernel which is well-know in the generalization of Wigner and cross-Wigner transforms (Gadella et al. 1991; Varilly 1989).

Our aim will be to generalize this formula in the case the Weyl-Heisenberg group is replaced by a Lie group (satisfying some constraints in order for the formula to be well-defined). The generalized formula will be:

$$\langle\hat{A}\rangle_{\psi_f\psi_i} = \frac{1}{\lambda^2} \int_G d\mu(g) \; Tr[\hat{A}U(g)] \; \frac{Tr[|\psi_i\rangle\langle\psi_f|U^\dagger(g)]}{\langle\psi_f|\psi_i\rangle} \tag{5}$$

where $d\mu(g)$ is the Haar measure on G (we suppose it unimodular, i.e. its left-invariant Haar measure is equal to its right-invariant Haar measure) and λ is a constant related to the dimension of the irreducible unitary and $\mathbb{C}$-linear representation U of G. The term $\frac{Tr[U^\dagger(g)|\psi_i\rangle\langle\psi_f|}{\langle\psi_f|\psi_i\rangle}$ can be identified to a quasi-distribution of probabilities (Brif and Mann 1998); (Abgaryan, Khvedelidze, and Torosyan 2019) whose values can be negative. If $|\psi_i\rangle$ is an admissible vector in the sense of the generalized coherent states (defined by groups acting on a very specific vector called *an admissible one*, see below), the term $Tr[U^\dagger(g)|\psi_i\rangle\langle\psi_f|] = \langle\psi_f|U^\dagger(g)|\psi_i\rangle$ takes the sense of a coherent state transform (which becomes wavelets transform in some particular choice of group: describing translation and change of scale).We refer here to the book of S. t. Ali, J.-P Antoine, J.-P. Gazeau (Ali, Antoine, and Gazeau 2000). The term $Tr[\hat{A}U(g)]$ plays the role of a Weyl symbol of $\hat{A}$.

One can wonder why it would be amenable to rewrite the weak value $\langle\hat{A}\rangle_{\psi_f\psi_i}$ in these terms. The answer of such a question is that it allows us to perform its harmonic analysis connected with a G-symmetry. Let us note that the Grossman-Royer is not so easy to write when we pass from the Weyl-Heisenberg group to an arbitrary unimodular Lie group with a square-integrable representation. If you give a group, then knowing the irreducible and square integrable representation $G \longrightarrow EndG \;\; g \longrightarrow \;\; U(g)$, you can immediately write the weak value without having to build an analog of the Grossman-Royer operator if any.

We have to say that some attempts were made to get the formula we give here. The work of F. Antonsen (Antonsen 1998) is very interesting and inspiring, from this point of view, but the formula he proposed does not seem to be the right one (it differs by a hermitian conjugate, but is crucial to us).

Our formula could also be adapted in the case of a symmetric space described by a coset G/H, where H is a subgroup of G. In some interesting particular cases, this coset can be endowed with a Kählerian structure and thus could mimic a generalized phase-space as it is done in the geometric quantization. It is worth noting that if our vector $|\psi_i\rangle$ is an admissible vector invariant under the subgroup H, the coset G/H is nothing but a set of coherent states.

This formula leads finally to corollaries, one of them being a formula introducing a kind of Moyal product (Varilly and Gracia-Bonda 1989).

2. The main formula and its proof

In this section we have to use a lemma:

Lemma. Let $\mathfrak{H}$ be a Hilbert space and $\{|i\rangle, |j\rangle, |k\rangle, \ldots\}$ an infinitely countable orthogonal basis of $\mathfrak{H}$. Let also $\hat{A}$ be a linear (bounded) operator $\mathfrak{H} \longrightarrow \mathfrak{H}$. It is always possible to write

$$\hat{A} = \sum_{ij} \alpha^{ij} |i\rangle\langle j| \tag{6}$$

where $\alpha^{ij} \in \mathbb{C}$.

Indeed, let $|\psi\rangle$ be a ket of $\mathfrak{H}$. We have:

$$|\psi\rangle = \sum_{j} \psi^j |j\rangle \tag{7}$$

where $\psi^j = \langle j|\psi\rangle$. We can express $\hat{A}|j\rangle$ in terms of his components:

$$\hat{A}|j\rangle = \sum_{i} \alpha^{ij} |i\rangle \tag{8}$$

Thus we can write successively:

$$\begin{aligned} \hat{A}|\psi\rangle &= \sum_{j} \psi^j \sum_{i} \alpha^{ij} |i\rangle \\ &= \sum_{ij} \alpha^{ij} |i\rangle\langle j|\psi\rangle \\ &= \left(\sum_{ij} \alpha^{ij} |i\rangle\langle j| \right) |\psi\rangle \end{aligned}$$

And so we obtain, as expected:

$$\hat{A} = \sum_{ij} \alpha^{ij} |i\rangle\langle j|$$

Theorem. Let $|\eta\rangle$ and $|\varphi\rangle$ be two states of the system: $|\eta\rangle, |\varphi\rangle \in \mathfrak{H}$. Then one has:

$$\lambda^2 \langle \eta|\hat{A}|\varphi\rangle = \int_G d\mu(g)\; Tr[\hat{A}\; U(g)]\; Tr[\, |\varphi\rangle\langle\eta|\; U^\dagger(g)] \tag{9}$$

where $\mu(g)$ is the so-called *left-invariant Haar-measure*.

In order to prove this theorem, let us begin – it is important – by writing the *orthogonality relations* for these two states explicitly and clarify the context[1] :

$$\langle \hat{C}\eta | \hat{C}\eta' \rangle \; \langle \varphi' | \varphi \rangle = \int_G d\mu(g) \; \overline{\langle \eta'_g | \varphi' \rangle} \; \langle \eta_g | \varphi \rangle \tag{10}$$

where

$$\begin{aligned} |\eta_g\rangle &= U(g)|\eta\rangle \\ |\eta'_g\rangle &= U(g)|\eta'\rangle \\ \hat{C} &= \lambda \mathbb{1} \end{aligned}$$

and where $U(g)$ is a square-integrable representation of a locally compact Lie group G on $\mathfrak{H}$, with $\lambda > 0$. Moreover, $|\eta\rangle$ and $|\eta'\rangle$ must be *admissible kets*, i.e.

$$I(\eta) := \int_G d\mu(g) |\langle U(g)\eta | \eta \rangle|^2 = \int_G d\mu(g) |\langle \eta | U(g) | \eta \rangle|^2 < \infty \tag{11}$$

Starting from (10) we can write

$$\begin{aligned} \langle \lambda \mathbb{1}\eta | \lambda \mathbb{1}\eta' \rangle \; \langle \varphi' | \varphi \rangle &= \int_G d\mu(g) \; \langle \varphi' | U(g) | \eta' \rangle \; \langle \eta | U^\dagger(g) | \varphi \rangle \\ \lambda^2 \; \langle \eta | \eta' \rangle \; \langle \varphi' | \varphi \rangle &= \int_G d\mu(g) \; \langle \varphi' | U(g) \mathbb{1} | \eta' \rangle \; \langle \eta | U^\dagger(g) \mathbb{1} | \varphi \rangle \end{aligned}$$

Let us put, as a particular case, $|\eta'\rangle = |i\rangle$ and $|\varphi'\rangle = |j\rangle$ (two basis kets of $\mathfrak{H}$), and insert the resolution of the identity:

$$\begin{aligned} \lambda^2 \; \langle \eta | i \rangle \; \langle j | \varphi \rangle &= \int_G d\mu(g) \; \langle j | U(g) \left(\sum_k |k\rangle \; \langle k| \right) |i\rangle \; \langle \eta | U^\dagger \left(\sum_l |l\rangle \; \langle l| \right) |\varphi\rangle \\ &= \int_G d\mu(g) \; \sum_k \langle k | i \rangle \; \langle j | U(g) | k \rangle \; \sum_l \langle l | \varphi \rangle \langle \eta | U^\dagger(g) | l \rangle \end{aligned}$$

Two traces appear clearly on the right side ; so we may conclude :

$$\lambda^2 \; \langle \eta | i \rangle \; \langle j | \varphi \rangle = \int_G d\mu(g) \; Tr[|i\rangle\langle j| U(g)] \; Tr[|\varphi\rangle\langle \eta | U^\dagger(g)] \tag{12}$$

This formula expresses a property of the operator $|i\rangle\langle j|$. Using our lemma, we can generalize. Let α^{ij} be complex numbers for every pair i, j of indexes labelling the basis-kets of $\mathfrak{H}$; We can write successively :

$$\lambda^2 \; \langle \eta | \alpha^{ij} | i \rangle \; \langle j | \varphi \rangle = \int_G d\mu(g) \; Tr[\alpha^{ij} |i\rangle\langle j| U(g)] \; Tr[|\varphi\rangle\langle \eta | U^\dagger(g)]$$

$$\lambda^2 \; \langle \eta | \left(\sum_{ij} \alpha^{ij} |i\rangle \; \langle j| \right) |\varphi\rangle = \int_G d\mu(g) \; Tr[\left(\sum_{ij} \alpha^{ij} |i\rangle\langle j| \right) U(g)] \; Tr[|\varphi\rangle\langle \eta | U^\dagger(g)]$$

1. This theorem can be found in (Ali, Antoine, and Gazeau 2000), p.156.

The two pairs of brackets contain the general expression of the operator $\hat{A}$; so we have obtained the following :

$$\lambda^2 \langle\eta|\hat{A}|\varphi\rangle = \int_G d\mu(g)\ Tr[U(g)\hat{A}]\ Tr[|\varphi\rangle\langle\eta|U^\dagger(g)] \tag{13}$$

This formula enables us to obtain the *weak values* of the operator $\hat{A}$ by means of traces and of the representation $U(g)$. Indeed, taking $|\eta\rangle = |\psi_f\rangle$ and $|\phi\rangle = |\psi_i\rangle$ (respectively, post- and pre-selected states) and dividing by $\langle\psi_f|\psi_i\rangle$, we also obtain:

$$\lambda^2 \frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = \langle\hat{A}\rangle_{\psi_f\psi_i} = \int_G d\mu(g)\ Tr[U(g)\hat{A}]\ \frac{Tr[|\psi_i\rangle\langle\psi_f|U^\dagger(g)]}{\langle\psi_f|\psi_i\rangle} \tag{14}$$

which is nothing but (5).

3. Some corollaries

We enunciate the corollaries : the proofs are obvious, and we do not give them. Let us first introduce a new function: the so-called *generalised Weyl function* (gWf) of the operator $\hat{A}$:

$$W_{\hat{A}}(g)\ :=\ Tr[\hat{A}\ U(g)] \tag{15}$$

Corollary 1 We get the following relation between gWf and traces of operator. As above, such traces will play an important role:

$$\lambda^2\ Tr[\hat{A}\ \hat{B}^\dagger]\ =\ \int_G d\mu(g)\ W_{\hat{A}}(g)\ W_{\hat{B}}(g) \tag{16}$$

Corollary 2 Another relationship the gWf and traces of operators can be proved:

$$\lambda^2\ W_{\hat{A}}(g')\ =\ \int_G d\mu(g)\ W_{\hat{A}}(g)\ Tr[U(g)\ U^\dagger(g')] \tag{17}$$

It should be noted that the trace on the left-hand side plays the role of *reproducing kernel* for $W_{\hat{A}}(g)$ if we define:

$$\lambda^2\ K(g, g') =\ Tr[U(g)\ U^\dagger(g')] \tag{18}$$

then in this way we can write:

$$\int_G d\mu(g)\ K(g, g')\ W_{\hat{A}}(g)\ =\ W_{\hat{A}}(g') \tag{19}$$

Now a third formula, wich can be introduced by defining first a new product (similar to a Moyal-Product).

Corollary 3 Let F and G be two functions belonging to $L^2(G, d\mu)$. We put

$$(F \star L)(g)\ :=\ \int_G\int_G d\mu(g')\ d\mu(g'')\ \frac{1}{\lambda^4}\ \overline{Tr[Ug')\ U(g'')\ U^\dagger(g)]}\ F(g')\ L(g'') \tag{20}$$

We can establish the following:

$$W_{\hat{A}\hat{B}}(g)\ =\ (W_{\hat{A}}\ \star\ W_{\hat{B}})(g) \tag{21}$$

wich means that the gWf of the product of two functions is equal to $\star$ − *product* of the gWf of the functions.

Corollary 4. It is a particular case of (17) for $\hat{B} = \mathbb{1}$:

$$\lambda^2 \, Tr\hat{A} = \int_G d\mu(g) \, W_{\hat{A}}(g) \tag{22}$$

Corollary 5. It is another particular case of (13) for $\hat{A} = U^\dagger(g')$:

$$\lambda^2 \, W_{\hat{A}}(g') = \int_G d\mu(g) \, W_{\hat{A}}(g) \, Tr[U(g) \, U^\dagger(g')] \tag{23}$$

Corollary 6. Let H be the maximal compact subgroup of G, let $\Omega \in G/H$ $(G = \Omega\, H)$. Let also the two vectors $|\eta\rangle$ and $|\eta'\rangle$ be such that $U(h)|\eta\rangle = |\eta\rangle$ and $U(h)|\eta'\rangle = |\eta'\rangle$. Then we get :

$$\frac{\lambda^2}{Vol(H)} \langle\eta|\eta'\rangle \, \langle\varphi'|\varphi\rangle = \int_{G/H} d\mu(\Omega) \, (\overline{W(\eta', \varphi')(\Omega)} \, W(\eta, \varphi)(\Omega) \tag{24}$$

where

$$W(\eta, \varphi)(\Omega) := Tr[U^\dagger(\Omega)|\varphi\rangle\langle\eta|]U^\dagger(g)|\varphi\rangle = \langle\eta|U^\dagger(\Omega)|\varphi\rangle \tag{25}$$

and hence

$$\frac{\lambda^2}{Vol(H)} Tr(\hat{B}\hat{A}) = \int_{G/H} d\mu(\Omega) \, \overline{W_{\hat{B}^\dagger}(\Omega)} \, W_{\hat{A}}(\Omega) \tag{26}$$

4. Two examples, G = SU(2) and G = SO(3).

In this section we try to show that our formula (13) can be applied to two important groups in Physics; we prove in every case that, with a suitable choice of the measure μ, both left-hand and right-hand sides of the equation are truly equal.

4.1. The main formula and SU(2). Let G be an abstract group represented by $SU(2)$, i.e. there exists a morphism $U: \; G \longrightarrow SU(2) \quad g \longrightarrow U(g)$ and two conditions: $U(g_1 g_2) = U(g_1)U(g_2)$ and $U(g)U^\dagger(g) = U^\dagger(g)U(g) = \hat{I}$ (identity) for all $g, \; g_1, \; g_2 \; \in G$. We can write

$$U(g) = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \tag{27}$$

$$U^\dagger(g) = \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix} \tag{28}$$

And thus we have:

$$U(g)U^\dagger(g) = U^\dagger(g)U(g) = \begin{pmatrix} \alpha\overline{\alpha} + \beta\overline{\beta} & 0 \\ 0 & \alpha\overline{\alpha} + \beta\overline{\beta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

if and only if $\alpha\overline{\alpha} + \beta\overline{\beta} = 1$. This condition is fulfilled with:

$$\begin{aligned} \alpha &= x_1 + ix_2 \\ \beta &= x_3 + ix_4 \end{aligned} \tag{29}$$

where

$$\begin{aligned} x_1 &= cos\theta \\ x_2 &= sin\theta\ cos\phi \\ x_3 &= sin\theta\ sin\phi\ cos\psi \\ x_4 &= sin\theta\ sin\phi\ sin\psi \end{aligned} \tag{30}$$

Indeed, $\alpha\overline{\alpha} + \beta\overline{\beta} = (x_1^2 + x_2^2) + (x_3^2 + x_4^2) = 1$.
When G is *the group of rotations*, this definition seems to be very natural and we have $\theta \in [0, \pi]$, $\phi \in [0, \pi]$ and $\psi \in [0, 2\pi]$.
Let

$$U(g_1) = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\overline{\beta}_1 & \overline{\alpha}_2 \end{pmatrix}$$

and

$$U(g_2) = \begin{pmatrix} \alpha_2 & \beta_2 \\ -\overline{\beta}_2 & \overline{\alpha}_2 \end{pmatrix}$$

The product is

$$U(g_1)U(g_2) = \begin{pmatrix} \alpha_1\alpha_2 - \beta_1\overline{\beta}_2 & \alpha_1\beta_2 + \beta_1\overline{\alpha}_2 \\ -\overline{\beta}_1\alpha_2 - \overline{\alpha}_1\overline{\beta}_2 & -\overline{\beta}_1\beta_2 + \overline{\alpha}_1\overline{\alpha}_2 \end{pmatrix}$$

If we define $\alpha_3 = \alpha_1\alpha_2 - \beta_1\overline{\beta}_2$ and $\alpha_1\beta_2 + \beta_1\overline{\alpha}_2$, we see that

$$U(g_1)U(g_2) = \begin{pmatrix} \alpha_3 & \beta_3 \\ -\overline{\beta}_3 & \overline{\alpha}_3 \end{pmatrix} \tag{31}$$

That is exactly the same form as (27), as needed.
Our choice of the measure will be the so-called *Haar measure* :

$$d\mu(g) = \frac{1}{2\pi^2}\, sin^2\theta\ sin\phi\ d\theta\ d\phi\ d\psi \tag{32}$$

The vectors will be of the form

$$|\eta\rangle = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \tag{33}$$

$$|\varphi\rangle = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

where $\eta_i, \varphi_i \in \mathbb{C}$. Their scalar product $\langle\, .\, |\, .\, \rangle$ will be *represented* by the matrix product. Provided with the usual addition law of two vectors, the set of all these vectors forms a two-dimensional complex

Hilbert space $\mathfrak{H}$, whose basis vectors are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. However, are these vectors admissible in the sense of (11) ? Yes. Indeed, it is easy to write

$$|\langle\eta|U(g)\eta\rangle|^2 = |\overline{\eta}_1\alpha\eta_1 - \overline{\eta}_2\overline{\beta}\eta_1 + \overline{\eta}_1\beta\eta_2 + \overline{\eta}_2\overline{\alpha}\eta_2|^2$$

Clearly, the right-hand side is finite (being composed only of a sinus and a cosinus). The integration on $SU(2)$ is made of integrations between 0 and π or 2π, whose results are necessarily finite.
As $\hat{A}$, we choose:

$$\hat{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \tag{34}$$

where all $a_i \in \mathbb{C}$. We must ensure that, representing an observable, the matrix $\hat{A}$ is *hermitian*, i.e. $\hat{A}^\dagger = \hat{A}$, which can be written as follows:

$$\hat{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \hat{A}^\dagger = \begin{pmatrix} \overline{a}_1 & \overline{a}_3 \\ \overline{a}_2 & \overline{a}_4 \end{pmatrix}$$

Thus $a_1, a_4 \in \mathbb{R}$. Moreover, $a_2 = \overline{a}_3$ and (which is the same), $a_3 = \overline{a}_2$. So, $\hat{A}$ must be rewritten:

$$\hat{A} = \begin{pmatrix} a_1 & a_2 \\ \overline{a}_2 & a_4 \end{pmatrix} \tag{35}$$

Let us now compute all the detailed elements of the relation (13). Successively:

$$|\varphi\rangle\langle\eta| = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \begin{pmatrix} \overline{\eta}_1 & \overline{\eta}_2 \end{pmatrix} = \begin{pmatrix} \varphi_1\overline{\eta}_1 & \varphi_1\overline{\eta}_2 \\ \varphi_2\overline{\eta}_1 & \varphi_2\overline{\eta}_2 \end{pmatrix} \tag{36}$$

It should be noted that the product $|\varphi\rangle\langle\eta|$ has been *represented* here by the usual tensor product of two matrices.
Our task is now to establish the relevance of the formula (13). Let us compute separately her left-hand side and her right-hand side.

Left-hand side:

$$\begin{aligned} \lambda^2\langle\eta|\hat{A}|\varphi\rangle &= \lambda^2 \begin{pmatrix}\eta_1 & \eta_2\end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ \overline{a}_2 & a_4 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \\ &= \lambda^2(\overline{\eta}_1 a_1\varphi_1 + \overline{\eta}_2\overline{a}_2\varphi_1 + \overline{\eta}_1 a_2\varphi_2 + \overline{\eta}_2 a_4\varphi_2) \end{aligned} \tag{37}$$

Right-hand side
Successively :

$$U(g)\hat{A} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ \overline{a}_2 & a_4 \end{pmatrix} = \begin{pmatrix} \alpha a_1 + \beta\overline{a}_2 & \alpha a_2 + \beta a_4 \\ -\overline{\beta}a_1 + \overline{\alpha}\overline{a}_2 & -\overline{\beta}a_2 + \overline{\alpha}a_4 \end{pmatrix}$$

Note that we have *represented* the composed of the two operators $U(g)$ and $\hat{A}$ by their usual matrix product. We would insist on the fact that it is *a choice*. An other choice, for instance their tensor product, would have been possible. The ultimate justification of our choice and all the other choices we have made lies in the relevance of the formula (13) that we try to establish. Now:

$$Tr[U(g)\hat{A}] \;=\; \alpha a_1 + \beta \bar{a}_2 - \bar{\beta} a_2 + \bar{\alpha} a_4 \tag{38}$$

$$\begin{aligned} |\varphi\rangle\langle\eta|U^\dagger(g) &= \begin{pmatrix} \varphi_1\bar{\eta}_1 & \varphi_1\bar{\eta}_2 \\ \varphi_2\bar{\eta}_1 & \varphi_2\bar{\eta}_2 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix} \\ &= \begin{pmatrix} \varphi_1\bar{\eta}_1\bar{\alpha} + \varphi_1\bar{\eta}_2\bar{\beta} & -\varphi_1\bar{\eta}_1\beta + \varphi_1\bar{\eta}_2\alpha \\ \varphi_2\bar{\eta}_1\bar{\alpha} + \varphi_2\bar{\eta}_2\bar{\beta} & -\varphi_2\bar{\eta}_1\beta + \varphi_2\bar{\eta}_2\alpha \end{pmatrix} \end{aligned}$$

$$Tr[|\varphi\rangle\langle\eta|U^\dagger] \;=\; \varphi_1\bar{\eta}_1\bar{\alpha} + \varphi_1\bar{\eta}_2\bar{\beta} - \varphi_2\bar{\eta}_1\beta + \varphi_2\bar{\eta}_2\alpha \tag{39}$$

On the right-hand side of (13), the integrand is a product of three factors:

$$d\mu(g) \;\times\; Tr[U(g)\hat{A}] \;\times\; Tr[|\phi\rangle\langle\eta|U^\dagger(g)]$$

If we are explicit, we must write by introducing (29) and (30):

$$\begin{aligned} & d\theta\; d\phi\; d\psi\; \frac{1}{2\pi^2}\; sin^2\theta\; sin\phi \\ \times \Big(& a_1(cos\theta + sin\theta\; cos\phi) + \bar{a}_2(sin\theta\; sin\phi\; cos\psi + i\; sin\theta\; sin\phi\; sin\psi \\ & -a_2(sin\theta\; sin\phi\; cos\psi - i\; sin\theta\; sin\phi\; sin\psi) + a_4(cos\theta - i\; sin\theta\; cos\phi)\Big) \\ \times \Big(& \varphi_1\bar{\eta}_1(cos\theta - i\; sin\theta\; cos\phi) + \varphi_1\bar{\eta}_2(sin\theta\; sin\phi\; cos\psi - i\; sin\theta\; sin\phi\; sin\psi \\ & -\varphi_2\bar{\eta}_1(sin\theta\; sin\phi\; cos\psi + i\; sin\theta\; sin\phi\; sin\psi) + \varphi_2\bar{\eta}_2(cos\theta + i\; sin\theta\; cos\phi)\Big) \end{aligned}$$

Each pair of big brakets contains 8 terms; thus the product contains 64 terms, but unfortunately it is not useful to write each of them. Why? In the course of the triple integration, many of them will give a null contribution. The reason lies in the following definite integrals (which all clearly occur in the integration process).

$$\int_0^\pi d\theta\; sin^3\theta\; cos\theta = 0$$

$$\int_0^\pi d\phi\; cos\phi \;=\; \int_0^\pi d\phi\; cos\phi\; sin\phi \;=\; 0$$

$$\int_0^{2\pi} d\psi\; cos\psi \;=\; \int_0^{2\pi} d\psi\; sin\psi \;=\; \int_0^{2\pi} d\psi\; sin\psi\; cos\psi \;=\; 0$$

The only terms of the integrand that contribute to the final value of the right-hand side of (13) can now be written:

$$\begin{aligned} d\theta\; d\phi\; d\psi\; \frac{1}{2\pi^2}\; sin^2\theta\; sin\phi\; \Big(& a_1\varphi_1\bar{\eta}_1(cos^2\theta \;+\; sin^2\theta\; cos^2\phi) + a_1\varphi_2\bar{\eta}_2(cos^2\theta - sin^2\theta\; cos^2\phi) \\ & +\bar{a}_2\varphi_1\bar{\eta}_2(sin^2\theta\; sin^2\phi\; cos^2\psi \;+\; sin^2\theta\; sin^2\phi\; sin^2\psi) \\ & -\bar{a}_2\varphi_2\bar{\eta}_1(sin^2\theta\; sin^2\phi\; cos^2\psi - sin^2\theta\; sin^2\phi\; sin^2\psi) \\ & -a_2\varphi_1\bar{\eta}_2(sin^2\theta\; sin^2\phi\; cos^2\psi \;-\; sin^2\theta\; sin^2\phi\; sin^2\psi) \\ & -a_2\varphi_2\bar{\eta}_1(sin^2\theta\; sin^2\phi\; cos^2\psi + sin^2\theta\; sin^2\phi\; sin^2\psi) \\ & +a_4\varphi_2\bar{\eta}_2(cos^2\theta + \sin^2\theta\; cos^2\phi)\Big) \end{aligned}$$

The integration on ψ introduces a multiplicative factor 2π ; if we take into account that $cos^2\psi + sin^2\psi = 1$ and $cos^2\psi - sin^2\psi = cos2\psi$, the new integrand must now be rewritten as

$$
\begin{aligned}
d\theta\, d\phi\, \frac{1}{2\pi^2}\, sin^2\theta\, sin\phi \;\times\; 2\pi\Big(& a_1\varphi_1\bar{\eta}_1(cos^2\theta + sin^2\theta\, cos^2\phi) + a_1\varphi_2\bar{\eta}_2(cos^2\theta - sin^2\theta\, cos^2\phi) \\
& +\bar{a}_2\varphi_1\bar{\eta}_2(sin^2\theta\, sin^2\phi) \\
& -\bar{a}_2\varphi_2\bar{\eta}_1(sin^2\theta\, sin^2\phi\, cos2\psi) \\
& -a_2\varphi_1\bar{\eta}_2(sin^2\theta\, sin^2\phi\, cos2\psi) \\
& -a_2\varphi_2\bar{\eta}_1(sin^2\theta\, sin^2\phi) \\
& +a_4\varphi_2\bar{\eta}_2(cos^2\theta + sin^2\theta\, cos^2\phi)\Big)
\end{aligned}
$$

The fourth and fifth terms of this sum contain $cos2\psi$. The integration on ψ from 0 to 2π gives $\int_0^{2\pi} d\psi\, cos2\psi = 0$. Therefore, these two terms do not contribute to the final result and we can rewrite the terms of the integrand who really contributes to the right-hand side of (13) as

$$
\begin{aligned}
& d\theta\, d\phi \times \\
& \frac{1}{\pi}\Big(a_1\varphi_1\bar{\eta}_1(sin^2\theta\, \cos^2\theta\, sin\phi + sin^4\theta\, cos^2\phi\, sin\phi) + a_1\varphi_2\eta_2(sin^2\theta\, cos^2\theta\, sin\phi - sin^4\theta\, cos^2\phi\, sin\phi) \\
& + \bar{a}_2\varphi_1\bar{\eta}_2\, sin^4\theta\, sin^3\phi + a_2\varphi_2\bar{\eta}_1 sin^4\theta\, \sin^3\phi \\
& a_4\varphi_1\bar{\eta}_1(sin^2\theta\, cos^2\theta\, sin\phi - sin^4\theta\, cos^2\phi\, sin\phi) + a_4\varphi_2\bar{\eta}_2(sin^2\theta\, cos^2\theta\, sin\phi + sin^4\theta\, cos^2\phi\, sin\phi)
\end{aligned}
$$

In order to provide the final result of the calculation, we need the following definite integrals:

$$
\begin{aligned}
\int_0^{\pi} d\theta\, sin^2\theta = \int_0^{\pi} d\phi\, sin^2\phi &= \frac{\pi}{2} \\
\int_0^{\pi} d\theta\, sin^4\theta &= \frac{3\pi}{8} \\
\int_0^{\pi} d\theta\, sin\theta &= 2 \\
\int_0^{\pi} d\phi\, sin\phi\, cos^2\phi &= \frac{2}{3} \\
\int_0^{\pi} d\theta\, sin^2\theta\, cos^2\theta &= \frac{\pi}{8} \\
\int_0^{\pi} d\phi\, sin^3\phi &= \frac{4}{3}
\end{aligned}
$$

Final result : the right-hand side of (13):

$$
\begin{aligned}
& \frac{1}{\pi}\Big(a_1\varphi_1\bar{\eta}_1(\frac{\pi}{8}\times 2 + \frac{3\pi}{8}\times\frac{2}{3}) + a_1\varphi_2\bar{\eta}_2(\frac{\pi}{8}\times 2 - \frac{3\pi}{8}\times\frac{2}{3}) + \bar{a}_2\varphi_1\bar{\eta}_2\times\frac{3\pi}{8}\times\frac{4}{3}) + a_2\varphi_2\bar{\eta}_1\frac{3\pi}{8}\times\frac{4}{3} \\
& + a_4\varphi_1\bar{\eta}_1(\frac{\pi}{8}\times 2 - \frac{3\pi}{8}\times\frac{2}{3}) + a_4\varphi_2\,\bar{\eta}_2(\frac{\pi}{8}\times 2 + \frac{3\pi}{8}\times\frac{2}{3})\Big) \\
& = \frac{1}{\pi}\Big(a_1\varphi_1\bar{\eta}_1\times\frac{\pi}{2} + a_2\varphi_1\bar{\eta}_2\times\frac{\pi}{2} + a_2\varphi_2\bar{\eta}_1\frac{\pi}{2} + a_4\varphi_2\bar{\eta}_2\times\frac{\pi}{2}\Big) \\
& = \frac{1}{2}\Big(a_1\varphi_1\bar{\eta}_1 + \bar{a}_2\varphi_1\bar{\eta}_2 + a_2\varphi_2\bar{\eta}_1 + a_4\varphi_2\bar{\eta}_2\Big)
\end{aligned}
$$

And thus we get

$$
\int_{SU(2)} d\mu(g)\, Tr[U(g)\hat{A}]\, Tr[|\varphi\rangle\langle\eta|U^{\dagger}(g)] = \frac{1}{2}\Big(a_1\varphi_1\bar{\eta}_1 + \bar{a}_2\varphi_1\bar{\eta}_2 + a_2\varphi_2\bar{\eta}_1 + a_4\varphi_2\bar{\eta}_2\Big) \qquad (40)
$$

This is exactly what we had obtained in (37), except for the multiplicative factor λ^2. The comparison between (37) and (40) tells us, as expected, that:

$$\lambda^2\langle\eta|\hat{A}|\varphi\rangle = \int_{SU(2)} d\mu(g)\ Tr[U(g)\hat{A}]\ Tr[|\varphi\rangle\langle\eta|U^\dagger(g)] \tag{41}$$

provided $\lambda^2 = \frac{1}{2}$ or

$$\lambda = \frac{1}{\sqrt{2}} \tag{42}$$

Furthermore we know that $dim\ \mathfrak{H} = 2$ (the dimension of the Hilbert space $\mathfrak{H}$ we have initially chosen). We may thus conclude that:

$$\lambda = \frac{1}{\sqrt{dim\ \mathfrak{H}}} \tag{43}$$

This last relation is in perfect agreement with the relation (8.49) of (Ali, Antoine, and Gazeau 2000): as far as $SU(2)$ is concerned, our goal is achieved.
Among all observables that are worth considering, are the square and the z-component of the spin, namely $\hat{S}^2 = (\hbar^2/4)\sigma^2$ and $S_z = (\hbar/2)\sigma_z$, for wich:

$$\begin{aligned}\sigma^2 &= 4\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 4\sigma_0\\ \sigma_z &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{aligned}$$

For the first, we have $a_1 = a_4 = (\hbar^2/4)$, $a_2 = 0$; for the second, $a_1 = a_4 = (\hbar/2)$, $a_2 = 0$. The weak values are easy to write:

$$\begin{aligned}\langle\eta|\hat{S}^2|\varphi\rangle &= \frac{\hbar^2}{4}(\bar{\eta}_1\varphi_1 + \bar{\eta}_2\varphi_2)\\ \langle\eta|\hat{S}_z|\varphi\rangle &= \frac{\hbar}{2}(\bar{\eta}_1\varphi_1 - \bar{\eta}_2\varphi_2)\end{aligned}$$

And the general formula (13) becomes:

$$\lambda^2\langle\eta|\hat{S}^2|\varphi\rangle = \int_{SU(2)} d\mu(g)\ Tr[U(g)\hat{S}^2]\ Tr[|\varphi\rangle\langle\eta|U^\dagger(g)]$$

$$\lambda^2\langle\eta|\hat{S}_z|\varphi\rangle = \int_{SU(2)} d\mu(g)\ Tr[U(g)\hat{S}_z]\ Tr[|\varphi\rangle\langle\eta|U^\dagger(g)]$$

A similar formula holds for $\hat{S}_y$ and $\hat{S}_z$, but not for $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$, who are not hermitian.

4.2. The main formula and SO(3). Now, let G be an abstract group represented by $SO(3)$ (For instance, such a group could be the group of rotations in three-dimensionnal space). In this case, $U(g)$ will be a 3x3 matrix belonging to $SO(3)$. Generally, we can note this matrix as follows:

$$U(g) = \begin{pmatrix}u_{11} & u_{12} & u_{13}\\ u_{21} & u_{22} & u_{23}\\ u_{31} & u_{32} & u_{33}\end{pmatrix} \tag{44}$$

We have written u_{ik} instead of $u_{ik}(g)$ for brevity. Recall that if $U(g) \in SO(3)$ the following relations hold:

$$det\ U(g) = 1 \tag{45}$$

$$U^T(g)U(g) = U(g)U^T(g) = I_3 \tag{46}$$

On the other hand, if g is a rotation, we can use the Euler's angles $\phi \in [0, 2\pi]$, $\theta \in [0, \pi]$, $\psi \in [0, 2\pi]$ and write:

$$U(g) = \begin{pmatrix} cos\psi & -sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which can be rewritten as

$$U(g) = \begin{pmatrix} cos\psi\ cos\phi - sin\psi\ cos\theta\ sin\phi & -cos\psi\ sin\phi - sin\psi\ cos\theta\ cos\phi & sin\psi\ sin\theta \\ sin\psi\ cos\phi + cos\psi\ cos\theta\ sin\phi & -sin\psi\ sin\phi + cos\psi\ cos\theta cos\phi & -cos\psi\ sin\theta \\ sin\theta\ sin\phi & sin\theta\ cos\phi & cos\theta \end{pmatrix} \tag{47}$$

This form is well known in group theory and we verify (45) and (46). Moreover, if $U(g_1)$ and $U(g_2)$ are orthogonal, have a determinant equal to 1 and represent the rotations g_1 and g_2, the product $U(g_1)U(g_2)$ is also orthogonal, has a determinant equal to 1, therefore belongs also to $SO(3)$ and is also of a form such as (47); it *represents* the rotation $g_1 g_2$: [2]

$$U(g_1)\ U(g_2) = U(g_1 g_2)$$

We can write the state vectors as follows:

$$|\varphi\rangle := \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \tag{48}$$

$$|\eta\rangle := \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} \tag{49}$$

where φ_i and $\eta_k \in \mathbb{C}$ for $i, k = 1, 2, 3$. Consequently it is natural to write:

$$|\varphi\rangle\ \langle\eta| := \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \otimes \begin{pmatrix} \overline{\eta}_1 & \overline{\eta}_2 & \overline{\eta}_3 \end{pmatrix} = \begin{pmatrix} \varphi_1\overline{\eta}_1 & \varphi_1\overline{\eta}_2 & \varphi_1\overline{\eta}_3 \\ \varphi_2\overline{\eta}_1 & \varphi_2\overline{\eta}_2 & \varphi_2\overline{\eta}_3 \\ \varphi_3\overline{\eta}_1 & \varphi_3\overline{\eta}_2 & \varphi_3\overline{\eta}_3 \end{pmatrix} \tag{50}$$

We have *represented* the product of the ket $|\varphi\rangle$ and the bra $\langle\eta|$ by the tensor product of two matrices. We must ensure that the kets $|\varphi\rangle$ and $|\eta\rangle$ are admissible, i.e. $|\langle\varphi|U(g)|\eta\rangle|^2$ is finite:

$$|\langle\varphi|U(g)|\varphi\rangle|^2 = |\begin{pmatrix} \overline{\varphi}_1 & \overline{\varphi}_2 & \overline{\varphi}_3 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} |^2$$

2. Such an affirmation can be found an proved in every standard book on Group Theory, v.g. W.M. Miller, *Symmetry Groups and their Applications*, Academic Press, New York and London, 1972.

That is effectively true because all terms of the matrix product are of the form $\overline{\varphi}_i u_{ik} \varphi_l$, which are clearly finite.
As observable, we choose an hermitian 3x3 complex matrix:

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{51}$$

with $a_{ik} = \overline{a}_{ki}$, $a_{ii} \in \mathbb{R}$. Let us write the detailed expressions that we must compute in order to verify the formula (13) in this particular case. First, on the left-hand side of (13): we use (51):

$$U(g)\hat{A} \; := \; \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{52}$$

To compute the trace, we need only to add the diagonal elements of the product:

$$Tr[U(g)\hat{A}] \; = \; (u_{11}a_{11} + u_{12}a_{21} + u_{13}a_{31}) + (u_{21}a_{12} + u_{22}a_{22} + u_{23}a_{32}) + (u_{31}a_{13} + u_{32}a_{23} + u_{33}a_{33}) \tag{53}$$

We proceed in the same way with :

$$|\varphi\rangle\langle\eta|U^{\dagger}(g) \; = \; \begin{pmatrix} \varphi_1\overline{\eta}_1 & \varphi_1\overline{\eta}_2 & \varphi_1\overline{\eta}_3 \\ \varphi_2\overline{\eta}_1 & \varphi_2\overline{\eta}_2 & \varphi_2\overline{\eta}_3 \\ \varphi_3\overline{\eta}_1 & \varphi_3\overline{\eta}_2 & \varphi_3\overline{\eta}_3 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}^T \tag{54}$$

Sum of diagonal elements:

$$\begin{aligned} Tr[|\varphi\rangle\langle\eta|U^{\dagger}(g)] \; = \; & (\varphi_1\overline{\eta}_1 u_{11} + \varphi_1\overline{\eta}_2 u_{12} + \varphi_1\overline{\eta}_3 u_{13}) + (\varphi_2\overline{\eta}_1 u_{21} + \varphi_2\overline{\eta}_2 u_{22} + \varphi_2\overline{\eta}_3 u_{23}) \\ & +(\varphi_3\overline{\eta}_1 u_{31} + \varphi_3\overline{\eta}_2 u_{32} + \varphi_3\overline{\eta}_3 u_{33}) \end{aligned} \tag{55}$$

The left-hand term of (13):

$$\lambda^2\langle\eta|\hat{A}|\varphi\rangle \; = \; \lambda^2 \begin{pmatrix} \overline{\eta}_1 & \overline{\eta}_2 & \overline{\eta}_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

$$\begin{aligned} \lambda^2\langle\eta|\hat{A}|\varphi\rangle \; = \; & \overline{\eta}_1 a_{11}\varphi_1 + \overline{\eta}_2 a_{21}\varphi_1 + \overline{\eta}_3 a_{31}\varphi_1 + \overline{\eta}_1 a_{12}\varphi_2 + \overline{\eta}_2 a_{22}\varphi_2 + \overline{\eta}_3 a_{32}\varphi_2 \\ & + \overline{\eta}_1 a_{13}\varphi_3 + \overline{\eta}_2 a_{23}\varphi_3 + \overline{\eta}_3 a_{33}\varphi_3 \end{aligned} \tag{56}$$

Let us try to express the integrand on the right-hand side of (13):

$$\begin{aligned} d\mu(g) \; \times Tr[U(g)\hat{A}] \; \times Tr[|\varphi\rangle\langle\eta|U^{\dagger}(g)] = \\ d\theta \; d\phi \; d\psi \; \frac{1}{8\pi^2} \; sin\theta \\ \times\Big((u_{11}a_{11} + u_{12}a_{21} + u_{13}a_{31}) + (u_{21}a_{12} + u_{22}a_{22} + u_{23}a_{32}) + (u_{31}a_{13} + u_{32}a_{23} + u_{33}a_{33})\Big) \\ \times\Big((\varphi_1\overline{\eta}_1 u_{11} + \varphi_1\overline{\eta}_2 u_{12} + \varphi_1\overline{\eta}_3 u_{13}) + (\varphi_2\overline{\eta}_1 u_{21} + \varphi_2\overline{\eta}_2 u_{22} + \varphi_2\overline{\eta}_3 u_{23}) + \\ + (\varphi_3\overline{\eta}_1 u_{31} + \varphi_3\overline{\eta}_2 u_{32} + \varphi_3\overline{\eta}_3 u_{33})\Big) \end{aligned} \tag{57}$$

Where, following (44) and (47), we must use

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} = \begin{pmatrix} cos\psi\ cos\phi - sin\psi\ cos\theta\ sin\phi & -cos\psi\ sin\phi - sin\psi\ cos\theta\ cos\phi & sin\psi\ sin\theta \\ sin\psi\ cos\phi + cos\psi\ cos\theta\ sin\phi & -sin\psi\ sin\phi + cos\psi\ cos\theta cos\phi & -cos\psi\ sin\theta \\ sin\theta\ sin\phi & sin\theta\ cos\phi & cos\theta \end{pmatrix} \quad (58)$$

Fully developed, the expression (57) contains 81 terms to be integrated; they are of the form:

$$d\theta\ d\phi\ d\psi\ \frac{1}{8\pi^2}\ sin\theta\ u_{ij}u_{kl}\ \varphi_k\bar{\eta}_l$$

It seems to be a long and complicated task. Fortunately, it is possible, using *visually* (58) and using the tables of definite integrals presented above, to *see* easily that 72 terms vanish. The remaining 9 terms are easy to compute: all are equal to 1/3, and correspond exactly to the 9 terms of (56). As an example, we could compute:

$$\frac{1}{8\pi^2}\ sin\theta\ u_{13}^2\ a_{31}\varphi_1\bar{\eta}_3 = \frac{1}{8\pi^2}\ sin^3\theta\ sin^2\psi\ a_{31}\varphi_1\bar{\eta}_3$$

After integration:

$$\left(\frac{1}{8\pi^2} \times \frac{4}{3} \times \pi \times 2\pi\right) a_{31}\varphi_1\bar{\eta}_3 = \frac{1}{3}\ a_{31}\varphi_1\bar{\eta}_3$$

We could also compute:

$$\frac{1}{8\pi^2}\ sin\theta\ u_{31}u_{13}\ a_{13}\varphi_1\bar{\eta}_3 = \frac{1}{8\pi^2}\ sin^3\theta\ sin\phi\ sin\psi$$

After integration:

$$\left(\frac{1}{8\pi^2} \times \frac{4}{3} \times 0 \times 0\right) a_{13}\varphi_1\bar{\eta}_3 = 0$$

We may conclude:

$$\lambda^2\ \langle\eta|\hat{A}|\varphi\rangle = \int_{SO(3)} d\mu(g)\ Tr[U(g)\hat{A}]\ Tr[|\varphi\rangle\langle\eta|U^\dagger(g)] \quad (59)$$

provided $\lambda^2 = 1/3$, or

$$\lambda = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{dim\ SO(3)}} = \frac{1}{\sqrt{dim\ \mathfrak{H}}} \quad (60)$$

in agreement with (8.49) of (Ali, Antoine, and Gazeau 2000).
We have established our main formula (59) in the general case, i.e. for any hermitian operator such as $\hat{A}$ in (51). Among these operators, we can take the particular case of *projectors* $\hat{\Pi}_x$, $\hat{\Pi}_y$, $\hat{\Pi}_z$:

$$\hat{\Pi}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (61)$$

$$\hat{\Pi}_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (62)$$

$$\hat{\Pi}_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{63}$$

And we can take the two normalized vectors:

$$|\eta\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad |\varphi\,\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \tag{64}$$

But we can also choose the kets $|\eta\rangle = |\psi_f\rangle$ and $|\varphi\rangle = |\psi_i\rangle$, respectively post-selected and pre-selected states, in a process of measure during which the system is slightly perturbated. In this case, we obtain the weak values we can write in this way:

$$\frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = 3\int_{SO(3)} d\mu(g)\; Tr[U(g)\hat{A}]\; \frac{Tr[|\psi_i\rangle\langle\psi_f|U^\dagger(g)]}{\langle\psi_f|\psi_i\rangle} \tag{65}$$

4.3. The main formula and G/H. Let H be a normal subgroup of G, i.e. $gH = Hg$ for all $g \in G$. Let also $|\psi_i\rangle$ be an initial state such as

$$U(h)|\psi_i\rangle = |\psi_i\rangle \tag{66}$$

for all $h \in H$. Then we also have (U is unitary) $U^\dagger(h)|\psi_i\rangle = U^{-1}(h)|\psi_i\rangle = U(h^{-1})|\psi_i\rangle = |\psi_i\rangle$, because $h^{-1} \in$ H. So,

$$U^\dagger(h)|\psi_i\rangle = |\psi_i\rangle \tag{67}$$

Successively,

$$\begin{aligned}
\lambda^2\langle\psi_f|A|\psi_i\rangle &= \int_G d\mu(g)\; Tr[U(g)\hat{A}]\; Tr[|\psi_i\rangle\langle\psi_f|U^\dagger(g)] \\
&= \int_G d\mu(g)\; Tr[U(g)\hat{A}]\; \sum_\alpha \langle e_\alpha|\psi_i\rangle\langle\psi_f|U^\dagger(g)|e_\alpha\rangle \\
&= \int_G d\mu(g)\; Tr[U(g)\hat{A}]\; \sum_\alpha \langle\psi_f|U^\dagger(g)|e_\alpha\rangle\langle e_\alpha|\psi_i\rangle \\
&= \int_G d\mu(g)\; Tr[U(g)\hat{A}]\; \langle\psi_f|U^\dagger(g) \sum_\alpha \Big(|e_\alpha\rangle\langle e_\alpha\Big)|\psi_i\rangle \\
&= \int_G d\mu(g)\; Tr[U(g)\hat{A}]\; \langle\psi_f|U^\dagger(g)|\psi_i\rangle
\end{aligned} \tag{68}$$

We want to show now how to adapt the main formula (13) in the case where the group G is replaced by the coset G/H where H is an abelian maximal subgroup of G. Recall that every Lie algebra $\mathfrak{g}$ can be broken down into a Cartan subalgebra $\mathfrak{h}$ and another one $\mathfrak{p}$.

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \tag{69}$$

wich gives by exponentiation

$$G = H \otimes P \tag{70}$$

Here, $P = G/H$. Let us define $g \in G$ such as $g := hx$ with $h \in H$ and $x \in P = G/H$. One has

$$d\mu(g) = d\mu(h)\, d\mu(x) \tag{71}$$

We may rewrite :

$$\lambda^2 \langle\psi_f|\hat{A}|\psi_i\rangle = \int_H d\mu(h) \int_{G/H} d\mu(x)\ Tr[U(h)U(x)\hat{A}]\ \langle\psi_f|U^\dagger(hx)|\psi_i\rangle \tag{72}$$

$$= \int_H d\mu(h) \int_{G/H} d\mu(x)\ Tr[U(hx)\hat{A}]\ \langle\psi_f|U^\dagger(x)U^\dagger(h)|\psi_i\rangle \tag{73}$$

Now, we must note that the proof of (13) has *not* used the hermiticity of the operator $\hat{A}$. This means that (13) is true whether $\hat{A}$ is hermitian or not: $\hat{A}$ does not need to be observable. Thus the relation (68) is true for every linear operator $\hat{A}$ (however, if we want to give meaning to the notion of weak value, then $\hat{A}$ must be hermitian !). Let us define $\hat{B} = \hat{A}U(h^{-1})$. We can write:

$$\lambda^2 \langle\psi_f|\hat{B}|\psi_i\rangle = \int_H d\mu(h) \int_{G/H} d\mu(x)\ Tr[U(hx)\hat{B}]\ \langle\psi_f|U^\dagger(x)U^\dagger(h)|\psi_i\rangle$$

$$\lambda^2 \langle\psi_f|\hat{A}U(h^{-1})|\psi_i\rangle = \int_H d\mu(h) \int_{G/H} d\mu(x)\ Tr[\hat{B}U(hx)]\ \langle\psi_f|U^\dagger(x)U^\dagger(h)|\psi_i\rangle \tag{74}$$

But (left-hand side), $U(h^{-1})|\psi_i\rangle = |\psi_i\rangle$ and (right-hand side), $U^\dagger(h)|\psi_i\rangle = |\psi_i\rangle$. Then:

$$\lambda^2 \langle\psi_f|\hat{A}|\psi_i\rangle = \int_H d\mu(h) \int_{G/H} d\mu(x)\ Tr[\hat{A}U(h^{-1})U(h)U(x)]\ \langle\psi_f|U^\dagger(x)|\psi_i\rangle \tag{75}$$

And

$$\lambda^2 \langle\psi_f|\hat{A}|\psi_i\rangle = \int_H d\mu(h) \int_{G/H} d\mu(x)\ Tr[\hat{A}U(x)]\ \langle\psi_f|U^\dagger(x)|\psi_i\rangle \tag{76}$$

We could define "*volume*" of H as the measure of H :

$$Vol(H) := \mu(H) = \int_H d\mu(h) \tag{77}$$

So, the weak value of $\hat{A}$ is given by

$$\frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = \frac{Vol(H)}{\lambda^2} \int_{G/H} d\mu(x)\ Tr[\hat{A}U(x)]\ \frac{\langle\psi_f|U^\dagger(x)|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} \tag{78}$$

It is possible to give another form to this formula. Indeed we have, if the $|k\rangle$ forms a basis of $\mathfrak{H}$:

$$\begin{aligned}\langle\psi_f|U^\dagger(x)|\psi_i\rangle &= \langle\psi_f|U^\dagger(x)\Big(\sum_k |k\rangle\langle k|\Big)|\psi_i\rangle \\ &= \sum_k \langle\psi_f|U^\dagger(x)|k\rangle\langle k|\psi_i\rangle \\ &= \sum_k \langle k|\psi_i\rangle\langle\psi_f|U^\dagger(x)|k\rangle \\ &= Tr[|\psi_i\rangle\langle\psi_f|U^\dagger(x)]\end{aligned} \tag{79}$$

Formula (78) becomes :

$$\frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = \frac{Vol(H)}{\lambda^2}\int_{G/H} d\mu(x)\; Tr[\hat{A}U(x)]\; \frac{Tr[|\psi_i\rangle\langle\psi_f|U^\dagger(x)]}{\langle\psi_f|\psi_i\rangle} \quad (80)$$

And the right-hand side of (80) contains only traces.

An interesting example is given by:

$$G = SO(3) \qquad H = SO(2) \qquad P = \frac{SO(3)}{SO(2)}$$

where $SO(3)/SO(2)$ is nothing but the sphere S_2.

5. Conclusion.

The main formula (13) and his extension to coset (72) are probably not entirely new (if we consider the theory of generalized coherent stated à la Ali-Antoine-Gazeau), but the formulae we have proved are useful in the quantum weak value context. Furthermore, relation (72) could be interesting if we want to consider quantum theory starting from the phase space, which is taken as homogeneous Kählerian manifolds. In this theory, formula (68) enables us to express the transition amplitudes. The generalization of formulae (13) and (68 - 76) in the cases of noncompact groups is not obvious because we need some square-integrable irreducible representation. But in certain cases it is possible. In particular in situations where the coset spaces are the so-called classical domains (deeply studied by Jean-Pierre Gazeau, (Gazeau 1989): $SO(4,2)/SO(4) \otimes SO(2)$, $SO(3,2)/SO(3) \otimes SO(2)$, $SO(2,2)/SO(2) \otimes SO(2)$, $SO(1,2)/SO(2)$. All these manifolds are kählerian manifolds. The wave functions are to be chosen as elements of Hilbert spaces of analytical functions of such domains and square integrable with an appropriate (Bergman) measure.

6. A general perspective on this work

The work on weak measures is done in the context of a joint Belgian research project (ARC, "Action de Recherche Concertée") gathering physicists (Y. Caudano, L.Ballestros, J.-P. Fréché) mathematicians (T. Carletti, W. Delongha), logicians and philosophers (B. Hespel, V. Degauquier) of the University of Namur (Naxys and Esphin Research Institutes), dedicated to the interpretations of weak values and measurements. The weak value of an observable of a system is obtained during a very weak interaction (implying as little perturbation as possible) with the system constrained by imposing a pre-selected state and a post-selected state. The weak values have strange behaviors (they can be complex and go sometimes outside the usual spectrum) and in some cases they are related to values predicted by Bohm theory (thus they are interesting when you are interested in the study of interpretations of Quantum Theory). Weak values and measurements addressed many philosophical questions because. For example (as it was considered in a thought experiment imagined by Wheeler of a modified two-slit experiment) if you are fixing photons in a preselected state and after a long time (when photons are already flying away the source!), you chose various post-selected states, all these choices change radically the way you are describing the past. How can we interpret this fact? It is difficult to admit that present is influencing the past. Is it maybe rather convenient to think that modification of present knowledge of the future can modify the way we are understanding the past. But there is here the place for a debate on the structure of time (see Thomas Hertogh, On the origin of time. Stephen Hawking's final theory, Penguin Books, 2023). Our personal work is also connected to applications of weak measurements in cosmology (see the seminal works of Brout, Englert and Spindel for example). Post-selected states correspond here to prescribed final state of the universe. Here the knowledge of a final state could influence the way one is telling the origin of the universe… This addressing many interesting philosophical questions. Technically, we have to consider here quantum theory in curved space-time. Without entering completely into this tough subject, we have modestly begun to tackle the problem of defining weak values in the context of a curved space-time. We have chosen to begin with the Wigner phase-space formalism of Quantum Theory (based on functions: Wigner and cross-Wigner transforms, Weyl symbols of operators, …) and adapt it to the case of curved phase-space (being non trivial Kähler manifolds). The paper presented here is to be considered in such a context and try to search a way to express a weak value in the case of phase-space endowing with Lie group symmetry. This could serve as toy-model to explore weak values in the curved phase-space context (see the important work of Maurice de Gosson, The Wigner Transform, World Scientific, 2017). Mathematically our result can also be considered in the field of harmonic analysis on Lie groups and as a way to perform the Lie group harmonic analysis of a weak value.

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