Kielnarowa Review (2024), **1**, 46–70

Kielnarowa REVIEW

ARTICLE

Negation of the Tsirelson's conjecture and genericity in quantum mechanics

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Abstract

We show that the negation of the Tsirelson's conjecture can be understood as, and in fact follows from, the ZFC - genericity of infinite sequences of the QM outcomes. We extend the result of Landsman that there holds the equivalence of the statistics of the entire sequences with the statistic of the single QM measurements in such sequences, over stratified infinite tensor products of the finite dimensional spaces. The stratification is based on uncomputable Turing classes and also on the Solovay genericity and the equivalence is now corrected by the appearance of certain self-adjoint operator. From studies on algorithmic randomness follows that these two properties are in a sense orthogonal one to the other for random sequences. We show two models of ZFC which indeed separate them. This is based on the classic results for Cohen and random forcings in set theory and also on Takeuti's early results on Boolean-valued models and quantum mechanics. This separation is the tool for showing that not all correlations between commuting operators on infinite dimensional Hilbert spaces can be reproduced on a tensor product of two spaces with the corresponding self-adjoint operators. We discuss the findings also from the perspective of eventual practical applications. The supplementary subsections contain discussion of QM with the set theory component along with Solovay randomness of the outcomes.

Keywords: Tsirelson's conjecture, Boolean-valued models ZFC, quantum mechanics

1. Introduction

This paper concerns the formal relationship obtaining between three research domains: one in quantum mechanics (Tsirelson's conjecture), the second in formal set theory (the existence of generic sets for the models of set theory), and the third pertaining to the algorithmic randomness of binary infinite sequences (Turing classes of uncomputability and randomness). The results, though concerned with exploring the formal side of QM and so themselves being formal in character, may hopefully contribute to a better understanding of the future possibilities for new experimental research directions. QM on infinite dimensional Hilbert spaces, and the spaces of infinite sequences of QM outputs, belong to a particular domain open for eventual extensions, e.g. ref. Coladangelo and Stark 2020. Needless to say, reconciliation with general relativity (GR) constitutes a particular task, in which such extensions may find a place, and the recent proof negating Tsirelson's conjecture furnishes a motivation for looking at infinite constructions in QM more carefully. The infinite constructions in quantum physics are rather typical and belong to the heart of quantum field theory (QFT), statistical physics, and also quantum mechanics. They usually take the form of infinite, even uncountable, tensor products of Hilbert spaces (ITP) or operators, as in the case of QFT at any point of space-time where quantum fields are defined one has, formally, infinitely many interacting quantum systems. Another example is a thermodynamical limit, where what results are infinitely many interacting quantum particles or modes, and where the case also refers to ITPs. Infinite spin chains are subsets of ITPs, while Waveguide and Resonator Quantum Electrodynamics, e.g. ref. Heuck, Jacobs, and Englund 2020, also refers to ITPs when describing infinitely many transmission lines and modes in cavities. This paper indicates that there is a gap to be investigated when one extends the standard QM formalism over infinite constructions on the one hand, and considers quantum field theory, where these infinities are just there, on the other. This gap can be regarded as furnishing a reason for the incompatibility of QM and GR that so far remains unvanquished.

Tsirelson's problem is a conjecture to the effect that the sets of finitely many correlations of independent measurements of commuting observables on a Hilbert space $\mathcal{H}$ of a quantum system will always be reproducible by the complemented set of all finitely many independent measurements on the joint system with the product Hilbert space $\mathcal{H}_a \times \mathcal{H}_b$. (See the more precise description in the Key Terminologies and Results sections.) This has been recently shown to be false Zhengfeng Ji et al. 2022: i.e. the product case does not reproduce all correlations on the entire, necessarily infinite-dimensional, Hilbert space. Tsirelson's conjecture is known to be equivalent to the Connes embedding problem for operator algebras, already stated in 1976 by Alain Connes (Connes 1976; N. Ozawa 2013). Thus resolving Tsirelson's conjecture in the negative serves to resolve the Connes embedding problem.

The genericity problem in formal set theory is the question of the existence of generic filters G in models of Zermelo Fraenkel set theory with, eventually, the axiom of choice. For first-order theories with countable languages there will always exist countable models that, in the context of set theory, guarantee that generic filters do exist. This, however, means that the countable transitive model M is nontrivially extended into the forcing extension model M[G]. There are two basic forcing procedures in set theory, which are the random and Cohen forcings, and this last, around about 1963, led Peter Cohen (the inventor of the forcing procedure in mathematics) to independence proofs of the continuum hypothesis and axiom of choice from the Zermelo-Fraenkel axioms. Since then, various forcings have been construed over the years, leading to a tremendous richness in respect of independence results and set theory constructions. It has also been proved that generic filters do not exist for the universe (cumulative hierarchy) of sets V.

In algorithmic randomness theory, various classes of Turing uncomputability and notions of the randomness of infinite binary sequences are investigated in depth, and a subtle relation between random, and non-random though uncomputable, binary sequences has been worked out. In particular, some random sequences arise as being based on random genericity reduced to arithmetic, while some others do so as being based on Cohen genericity also reduced to arithmetic, and certain limiting rules (exclusion laws) hold between the two kinds, e.g. ref. Downey and Hirschfeldt 2010.

Tsirelson's problem in QM, and the genericity problem in set theory, even if they seem to be entirely separate, are in fact tightly connected in QM, and bridged by the direct generalization of the above limiting rules for the sequences. The way in which this comes about is the main concern of this work.

Even though the methods applied resemble, to some degree, the methods in the original proof of the negation of Tsirelson's conjecture ($\neg$TC) in ref. Zhengfeng Ji et al. 2022, the focus here is rather on understanding which infinite binary sequences, under general suppositions, might distinguish TC from $\neg$TC, rather than analysing the uncomputability and complexity of the entire sets of the correlations.

The objective of this work might seem technical, or even not directly related to quantum physics. However, we think the contrary is true, especially when one considers certain unsolved central problems of physics such as the reconciliation of quantum mechanics with general relativity, or some others that could become quite important in the not-so-distant future. The recognition of new limitations that will presumably be assigned to future working quantum computers is one such problem. This is related to the notions of randomness and classes of uncomputability that would characterize the new limits for quantum computations. True random sequences of numbers are considered limits for classical computers, as only pseudo-random numbers lie within reach for them, whereas these true random sequences could be realized by future quantum computers, e.g. Orts et al.Orts et al. 2023. Thus, a natural question is whether there are new limitations to this quantum randomness. More precisely, what is the hierarchy for quantum randomness envisaged by different QRNGs, and what is the common limiting randomness which would transcend all quantum random sequences but still remain within the scope of QM? Even though such questions seem to be of purely theoretical interest at present, they may serve to bootstrap the entire effort of seeking to understand ways in which QM could be extended. This could prove important when searching for an appropriate formalism such as would be required by a reconciliation of QM with GR. We are not deciding here that such limits are now in theory well defined, or capable of being accessed in the laboratory. Nevertheless, Tsirelson's conjecture, especially in terms of how it is formally negated, furnishes a motivation for initiating theoretical work on the topic of their accessibility. This is somewhat similar to considering extensions of QM in terms of Bell-style bounds, where no-signaling extensions are considered with bounds exceeded by $2\sqrt{2}$, e.g. ref. Masanes, Acin, and Gisin 2006. As the analysis in this paper shows, the infinite families of quantum measurements can amount to a delicate formal tool that enables one to distinguish TC from $\neg$TC on the one hand, and an understanding of randomness in QM on the other. However, this requires much care and insight when defining and manipulating such infinite constructs from different branches of mathematics. Thus, an understanding of Tsirelson's problem points to the formal side of QM that might be useful in respect of the task of its proper extension over infinite constructs. One deep result is that the Born probabilistic rule for a single measurement defines the probabilistic measure over the spaces of infinite sequences of outcomes of QM experiments, and determines the 1-randomness of the sequences (e.g. Landsman 2020 and Subsection 4.4). We believe that the problems with the extension of QM over infinities of modes or particles, as in quantum field theory, also require careful reconsideration in line with the methods adopted in this paper. This connection with quantum field theory and general relativity will be the topic of a separate forthcoming work by the present authors.

Let $\sigma_0 \in 2^{<\omega}$ be a binary finite sequence of outcomes of the repeating quantum mechanical yes-no measurements on the 2-dimensional Hilbert space $\mathcal{H}^{(2)} \simeq \mathbb{C}^2$, where $\mathbb{C}$ is the field of the complex numbers. Such a sequence is realized, e.g., when measuring the $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Pauli matrix in the normalized state $1/\sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, but the quite similar 'classical' process of tossing a symmetric coin cannot in fact be realized deterministically and so classically. Such finite repetitions can be extended towards infinite binary sequences $\sigma \in 2^{\omega}$ of the outcomes of repeating quantum measurements (e.g., ref. Landsman 2020). One could wonder whether there is any sense to such an extension up to infinity of the number of potential experiments considered within QM. On the one hand, this can be seen as a theoretical construct allowing one to test the borders of the applicability of the finite dimensional Hilbert spaces and the finite numbers of eigenfunctions and eigenvalues which are typically accessible within type I von Neumann algebras of observables. On the other hand, the infinite-dimensional cases are the part of conventional QM on Hilbert spaces, and the Born rule for a single run of quantum measurement already contains information about the probability of the whole run of infinite repetitions of the measurements. Thus, technically, the infinite case is already present in the standard formalism of QM.

The measurement is now performed on the entire Hilbert space $\mathcal{H} = (\mathcal{H}^{(2)})^K = \bigotimes_{i=0}^{K} \mathcal{H}_i^{(2)}$ in the $K < \infty$ repetitions case, or $\mathcal{H} = (\mathcal{H}^{(2)})^{\omega} = \otimes_{i\in\omega}\mathcal{H}_i^{(2)}$ for the infinite sequences of outcomes. The important observation made in ref. Landsman 2020 is that there is the equivalence (I_{AB}) between the following two procedures

A. s is generated in measurement performed as the 'whole run system' with $H^{\infty} = \bigotimes_{i=1}^{\infty} \mathcal{H}_i^{(2)}$, and then the statistical results on the ensemble of ss determine the probability measure P^{∞} on $\mathcal{H}^{\infty}$. One considers the 'whole run system' a quantum system on which there are performed measurements.
B. s is retrieved by collecting the statistical results at each ith, i.e. performed on $\mathcal{H}_i^{(2)}$, and thus drawing conclusions about the limiting statistical probability of the sequences in 2^{ω}.

As different as these situations may seem, they are nevertheless equivalent. This, again, is basically due to Landsman 2020, Theorem 5.1. The point is the Born rule, namely the fact that P^{∞}, is precisely P^{∞}_{Born} which, in turn, is uniquely generated by the single Born probability $P_{i,Born}$ Landsman 2020; Król, Bielas, and Asselmeyer-Maluga 2023. This point is also addressed in the supplementary subsection 4.4, where quantum states are explicitly referred to. We want to explore this equivalence further from the computational complexity point of view. Our aim is to obtain certain classes of infinite binary sequences which might distinguish two Hilbert spaces in QM, and the correlations of observables on them, which appear in the formulation of Tsirelson's conjecture (TC). So at first we claim that *there exist infinite binary sequences on infinite dimensional Hilbert spaces capable of distinguishing the correlations as in Tsirelson's conjecture on* $\mathcal{H}^{(\infty)}$ *and* $\mathcal{H}_A \otimes \mathcal{H}_B$.

Let us turn again to the equivalence I_{AB} and the infinite tensor product (ITP) of finite-dimensional Hilbert spaces $\mathcal{H}^{(\infty)}$. There is one general point connected with the ITP of finite-dimensional Hilbert spaces, which even for the minimal complex case $\dim(\mathcal{H}_i) = 2, i = 0, 1, 2, \dots$, leads to a non-separable Hilbert space. This has already been nicely analysed by John von NeumannVon Neumann 1939 (see also Thiemann and Winkler 2001), who showed that the Hilbert space $\mathcal{H}^{(\infty)}$ assigned to the countably infinite tensor product $\mathcal{H}^{(\infty)} = \bigotimes_{i=1}^{\infty} \mathcal{H}_i^{(2)}$ must be a non-separable infinite-dimensional one: the (complete) infinite tensor product of countably infinitely many, at least 2-dimensional, Hilbert spaces is *nonseparable* infinite-dimensional $\mathcal{H}^{(\infty)}$. However, as is shown by Landsmann, this does not affect the equivalence I_{AB}, which remains true Landsman 2020. One might think that the ordering in the sequences of Hilbert spaces matters, but it has also been shown by von Neumann that the resulting $\mathcal{H}^{(\infty)}$ does not depend on any ordering of infinitely countable sets of indices over which the tensoring is taken.

Thus we adopt a different perspective on the sets of indices, such that various Turing uncomputability classes might distinguish them. At the basic level, the set I of indices is to be *recursively enumerable* (r.e.), thus leading to r.e. sequences of spaces and the Hilbert space $\mathcal{H}^{(\infty)}$. The uncomputable power of sequences of indices is then enhanced upwards to the higher uncomputability Turing classes, thus enriching the structure of the infinite tensor products of Hilbert spaces. Computationally, there are different ITPs, $\text{ITP}_{\mathbf{a}}$ being where $\mathbf{a}$ is the Turing class of the sequence of indices $\sigma = (i_1, i_2, \dots, i_k, \dots), k = 1, 2, \dots$. The resulting infinite-dimensional Hilbert spaces $\mathcal{H}_{\mathbf{a}}^{(\infty)}$ are again non-separable, as whenever there exists a countable base for $\mathcal{H}$ there will exist a r.e. procedure for generating its subspaces spanned on the subsets of the vectors from a base, and the entire $\mathcal{H}$. Again, any direct application of the Turing classes to the sets of indices will not affect the procedure of ITP, nor the equivalence I_{AB}. One can also consider other definitions of ITPs, such as the inductive limit of finite-dimensional Hilbert spaces (A. Guichardet, 1969) or the approach involving grounded products of Hilbert spaces (J. Baez, 1993). These ITPs leads directly to separable Hilbert spaces, and from that point of view are suitable for certain physical applications; see also Subsection 4.4.

Next we will be allowing for random binary sequences of indices, and studying their influence on the ITP construction. This should be considered a purely theoretical tool, without imposing at this stage any observable effects. Even though one could not expect a big departure from the cases discussed so far, there are certain limitations to the applicability of random sequences together with all sequences from higher Turing classes, and the limitations are analysed in algorithmic randomness theory (e.g. ref.Downey and Hirschfeldt 2010). The obstructions appear in a pure form in higher random and higher Turing degrees. Instead of the word 'generic' that has usually figured in the algorithmic randomness literature (e.g. ref. Downey and Hirschfeldt 2010), we will be using the expression '(C)-generic' (Cohen generic) here, while 'randomness' will be understood as Martin-Löf (ML) algorithmic randomness (n-randomness, $n = 1, 2, \cdots$), e.g. ref. Downey and Hirschfeldt 2010. There is a set of important results connecting both kinds of infinite binary sequence: i.e. (C)-generic and random. The point is that the sequences are orthogonal in a special sense, in the entirety of Turing classes above 2-random and 2-generic. This is precisely the content of the following results:

Theorem 1 (Nies, Stephan, and Terwijn 2005) *Any 2-random sequence and any 2-(C)-generic sequence always form a minimal pair in the Turing degrees.*

We say that a Turing degree $\mathbf{b}$ bounds $\mathbf{a}$ when $\mathbf{a} \leq \mathbf{b}$ and this is when $\forall_{\sigma_{\mathbf{a}} \in \mathbf{a}} \forall_{\sigma_{\mathbf{b}} \in \mathbf{b}} \sigma_{\mathbf{a}} \leq_T \sigma_{\mathbf{b}}$, where $\leq_T$ is the Turing order relation between sets $\sigma_{\mathbf{a}}, \sigma_{\mathbf{b}}$. *A minimal pair* $(\mathbf{a}, \mathbf{b})$ of Turing degrees $\mathbf{a}$ and $\mathbf{b}$ for the random and generic sequences, is when the only degree $\mathbf{c} < \mathbf{a}$ and $\mathbf{c} < \mathbf{b}$ in the c.e. class is $\mathbf{c} = \mathbf{0}$. Then it holds

Theorem 2 (Kurtz 1981, Kautz 1991) *Every 2-random degree bounds a 1-(C)-generic degree.*

So, in the lower degrees, where $n \leq 2$, the randomness and (C)-genericity mix correspondingly, as in Theorem 2 above. For higher degrees, however, over and above 2-degree, an n-generic sequence will never be n-C-random and an n-random sequence will never be n-(C)-generic where $n > 2$, as follows from Theorem 1. This means that there is no nontrivial 'intersection' of $\mathbf{a}$ and $\mathbf{b}$ in the partial order of degrees. The above relation of random and (C)-generic sequences and their orthogonality in higher degrees offers a clue in respect of our search for tools capable of distinguishing TC and $\neg$TC.

There are, in general, two notions of genericity characterizing arithmetical and random sequences (e.g. refs. Downey and Hirschfeldt 2010; Kautz 1991). One is Cohen genericity, which is related to Cohen forcing in set theory, and here it is applied to Peano arithmetic (PA), while arithmetic randomness is closely related to Solovay genericity, which is also the specialization of the *random forcing* known from set theory to PA. Such specialization of set theory forcings to arithmetic is called *miniaturisation* Downey and Hirschfeldt 2010; Kautz 1991 and is based on ideas in ref. Fefferman 1964. Here we are performing a kind of *deminiaturisation*, which relies on extending the arithmetical perspective to set theory (ZFC) and relating this to QM on infinite dimensional Hilbert spaces (e.g. ref. Król, Bielas, and Asselmeyer-Maluga 2023). In particular, the arithmetic orthogonality of C-generic and random sequences is elevated to Cohen and random forcings in a certain model M of ZFC. It is known that given the forcing random extension $M[r]$ (where r represents a random generic real number), it does not contain any generic Cohen sequences (reals) amongst the random generics, and the converse also holds true. Based on the formal structure of the lattice of projections $\mathbb{L}(\mathcal{H}^{(\infty)})$ on the infinite-dimensional separable Hilbert space, one can infer that the structure will favor random forcing under certain conditions. Thus applying this bipartite exclusion principle of randomness and genericity for M and making use of some Boolean-valued model theory, one can find distinctions between the correlations on the product $\mathcal{H}_A \otimes \mathcal{H}_B$ from those on $\mathcal{H}$. It follows that there exist some correlations on $\mathcal{H}^{\infty}$ predicted by QM which can not be reproduced from those on $\mathcal{H}_A \otimes \mathcal{H}_B$.

Let us now briefly discuss the main findings of this work. 1. Given the equivalence I_{AB} from the beginning of this section, we show that it is actually stratified for QM on infinite dimensional Hilbert spaces – something which will depend on our way of constructing the infinite tensor product, specifically allowing for ZFC-generalized sequences of Hilbert spaces and observables. When one performs the measurements on the whole-run-system of such ZFC-generalized products, and compares this with the outcomes obtained 'locally' on the ith entry of the product, the results in the two cases can be statistically different. We find that in ZFC-generalized ITP (random generic) products there has to appear, 'generically,' a self-adjoint operator A that locally is absent from the collecting data. The operator A envisages a distinction between collecting local statistical outcomes and measurements on the whole-run-system on $\mathcal{H}^{\infty}$. Based on this, we can distinguish TC from $\neg$TC. On the other hand, local statistical information derived from the sequence (at ith entry) cannot distinguish TC and $\neg$TC, but rather reproduces the TC paradigm. This is not any direct negation of the equivalence I_{AB}; rather we are indicating generalized infinite tensor products (which are not the von Neumann ones) that differ with respect to such statistical predictions. 2. We carefully distinguish between the sequences of outcomes on $\mathcal{H}_a \otimes \mathcal{H}_b$ and on $\mathcal{H}^{\infty}$, showing directly, in particular, that the latter can be more correlated than the former. The finite refinement of this leads to the negation of TC. 3. We thus extract (following ref. Król, Bielas, and Asselmeyer-Maluga 2023) the appropriate form of randomness for infinite sequences on $\mathcal{H}^{\infty}$, which is ZFC-genericity (or Solovay set theory genericity) as distinct from pure arithmetic genericity (miniaturization to PA). QM (on infinite dimensional Hilbert spaces) leads to ZFC genericity, which requires the formal perspective of a model of ZFC. We consider this the fundamental feature of QM. 4. We discuss the findings from the perspective of their eventual applicability or practical use. We also discuss the problem of the eventual detection of ZFC-generic random sequences of QM outcomes and distinguishing these from arithmetic generic ones, where in practice this could lead to distinguishing TC from $\neg$TC.

The paper is organized in such a way that the central result is Theorem 4 in the Results section; however, its proof contains several stages encapsulated in lemmas, proposition and remarks that involve a presentation and discussion of set theory genericity in QM along with the above-mentioned issues. We close the main body of the paper with a Discussion and Supplementary Material section including various aspects of the extended QM formalism over set-theory component, measurements in such extended QM, ITPs, and Subsection 4.1 contains the proof of Proposition 1, which is a vital part of the proof of Theorem 4. The next section is the Key Terminologies section. The approach developed here mimics to some degree the Gödel incompleteness theorem: for sufficiently rich theories having ω-enumarable sets of axioms, we will always encounter some true sentences that remain unprovable in such axiomatized theories. The point is that when one takes higher than r.e. classes of axioms, the proofs can in general be attained and the 1st incompleteness theorem of Gödel does not hold. On our approach, we have the Turing enumerable tensor products of finite dimensional Hilbert spaces, which cannot approach sufficiently closely certain infinite-dimensional Hilbert spaces that can still figure in QM considerations. Then the whole-run-system measurements on Hilbert spaces like this lead to sequences allowing for a differentiation between product and nonproduct cases. The formal methods used here are therefore based on models of ZFC and, in particular, on the Boolean-valued models that, since the 1970s, have been developed in the context of QM by Gaisi Takeuti and Masanao Ozawa (e.g. refs. Takeuti 1978; M. Ozawa 2021).

2. Key terminologies

2.1 Turing uncomputable ITPs of Hilbert spaces

We refer Readers to ref. Soare 2016 for all additional information regarding Turing computability and uncomputable classes and to ref. Downey and Hirschfeldt 2010 regarding their relation to randomness. The supplementary subsection 4.4 also contains discussion of ITPs. Let $2^{<\omega}$ be a partial order of finite binary sequences and A a set. $A^{<\omega}$ is a set of all finite subsets of elements of A. Let A be some set of finite dimensional Hilbert spaces. Given $\sigma \in A^{<\omega}$ let $\bigotimes \sigma$ be $\bigotimes_i \sigma(i) = \bigotimes_{i \in [n]} \mathcal{H}_i$, $i = 1, 2, \ldots, n$; $\sigma = (\sigma_{j_i}, i = 1, 2, \ldots, n)$, $j_i \in \mathbb{N}$ and $\sigma_{j_i} = \sigma(i) = \mathcal{H}_i$ and $\dim(\mathcal{H}_i) = j_i$. Here $[n]$ is the finite sequence $[n] = (1, 2, \ldots, n)$. It certainly holds $\bigotimes \sigma = \mathcal{H}_\sigma$ is a finite dimensional Hilbert space and $\dim(\mathcal{H}_\sigma) = j_1 \cdot j_2 \cdot \ldots \cdot j_n$. We are interested in the process of attaining $\mathcal{H}^{\infty}$ an infinite dimensional Hilbert space, from finite dimensional Hilbert spaces in A. In general there are ITPs which govern this kind of successive up to infinity tensor products. This is considered as 'computationally trivial' in a sense that the sets of indices are Turing computable or computably enumerable (c.e.) e.g. ref. Soare 2016. Allowing for the uncomputable classes does not change much with the resulting ITPs (see Proposition 1 in the Results section). However, existence of such classes is the reason that not all sequences of Hilbert spaces are equally well-accessible from c.e. classes. Thus forming sequences of Hilbert spaces which are uncomputable, i.e. the sequences of their indices are represented by binary uncomputable sequences, and then tensoring collectively the Hilbert spaces with indices from this uncomputable sequence, is out of reach from trivial c.e. sequences of such spaces by *computable* processes. In principle, one could reach computationally such sequences by oracle Turing machines and oracles have to be 1-random (e.g. Martin Löf 1-random, ref. Downey and Hirschfeldt 2010) to guarantee an universal access to arbitrary sequences (such an 1-random oracle is uncomputable by itself)

Theorem 3 (Kučera 1984, Gács 1986) *Any sequence is computed by some 1-random sequence.*

However, by no means such 1-random sequence is unique for all sequences: there does not exist any single 1-random sequence computing all others. In contrary, there are many highly incomparable fractions of uncomputable sequences. In the Results section we will make use and elaborate over fraction of random vs. (C)-generic sequences and their mutual incompatibility (in higher degrees) in the context of the Tsirelson's conjecture.

Let us be more specific and as the example consider the spaces $2^{\omega} \times \mathcal{H}^{(2)}, 2^{<\omega} \times \mathcal{H}^{(2)}$ so thus they comprise sequences of 2-dimensional complex Hilbert spaces. If $\sigma \in 2^{\omega} \times \mathcal{H}^{(2)}, (2^{<\omega} \times \mathcal{H}^{(2)})$ then $\bigotimes \sigma$ is the infinite (finite) tensor product of the spaces. Even though the separable Hilbert spaces of the same dimension are isomorphic they can differ by the complexity of the sequences of the indices numbering the spaces. They are binary sequences belonging to 2^{ω} or $2^{<\omega}$. Consequently the spaces comprise the sequences belonging to different Turing classes, even though they number collection of identical Hilbert spaces like $\mathbb{C}^2$. Let $\mathbf{a}_s$ be the Turing class of a binary sequence s.

i. The tensor product $\bigotimes \sigma$ of a sequence $\sigma \in 2^{\omega} \times \mathcal{H}^{(2)}$ inherits the complexity of the binary sequence of its indices, $s \in 2^{\omega}$, and thus σ is in the same Turing class as s, i.e. $\mathbf{a}_{\sigma} \coloneqq \mathbf{a}_s$.
ii. An infinite dimensional Hilbert space $\mathcal{H}^{\infty}$ is attained in the Turing class $\mathbf{a}$ if there exist $\sigma \in A^{\omega}$ such that $\bigotimes \sigma = \mathcal{H}^{\infty}$ and $\mathbf{a}_{\sigma} = \mathbf{a}$.
iii. The pair $(\mathcal{H}^{\infty}, \sigma)$ is called $\mathbf{a}_{\sigma}$ $\mathcal{H}^{\infty}$ or $\mathcal{H}^{\infty}_{\mathbf{a}}$ Hilbert space. So $(\mathcal{H}^{\infty}, \sigma) = (\mathcal{H}'^{\infty}, \sigma')$ if not only $\mathcal{H}^{\infty} \overset{iso}{=} \mathcal{H}'^{\infty}$ as Hilbert spaces but also $\mathbf{a}_{\sigma} = \mathbf{a}_{\sigma'}$ in a sense that for any attainability class $\mathbf{a}_{\sigma}$ of $\mathcal{H}^{\infty}$ one finds equal to it the attainability class $\mathbf{a}_{\sigma'}$ of $\mathcal{H}'^{\infty}$ and conversely.

Remark 1 *In general, considering $\mathcal{H}$ as isometric to $\mathcal{H}'$ does not require fixing the computational classes of the spaces. However, the pairs $(\mathcal{H}^{\infty}, \sigma)$ clearly respect the Turing classes $\mathbf{a}_{\sigma}$ within which $\mathcal{H}^{\infty}$ has been generated as $\otimes\sigma$. So, formally, Turing uncomputability classes can matter as additional parameter.*

It might seem that appearence of uncomputable sequences of Hilbert spaces does not contribute much into considerations where the computability questions are not in the main scope of the analysis. However, we will see that certain refinements of the uncomputable sequences (ZFC twist) already determine the limits to the Landsman's equivalence and shed light on the Tsirelson's Conjecture. In what follows in general we are using $\mathcal{H}^{\infty}$ for a separable infinite dimensional complex Hilbert space and $\mathcal{H}^{(\infty)}$ for nonseparable infinite dimensional complex ones (like in the von Neumann's complete ITP). We are using the symbol σ as representing also binary sequences from 2^{ω} (not only the sequences of Hilbert spaces as above) but the use is clear from the context and should not lead to any misunderstanding.

2.2 The Tsirelson's Conjecture

The TC as QM problem can be expressed as follows ref. Scholz and Werner 2008 – more detail presentation is given in the Results section in the proof of Theorem 4. Let a, b be two independent observers conducting measurements over a quantum system S in $\mathcal{H}$ which is the Hilbert space of states of S. There are two possible scenarios realising the idea of independence of such independent measurements of a and b on $\mathcal{H}$. One scenario assigns $\mathcal{H}_a, \mathcal{H}_b$ two Hilbert spaces of states to a, b such that the joint system is described by $\mathcal{H}_a \otimes \mathcal{H}_b$ and clearly the measurements can be performed independently on each factor. However, noticing that independence enforces the commutativity of the observables of a with those of b, instead of factorizing of $\mathcal{H}$ into the tensor product one can perform measurements on the entire $\mathcal{H}$ under the supposition that the observables $\mathcal{A}$ and $\mathcal{B}$ commute as operators on $\mathcal{H}$. This is roughly the second scenario. The Tsirelson's problem (still roughly) is the statement deciding whether these two situations always lead to the same sets of correlation functions. TC says they are always equivalent, in a sense that given arbitrary $\mathcal{H}$ as in QM there always exist two $\mathcal{H}_a, \mathcal{H}_b$ and the 1st scenario on $\mathcal{H}$ is always equivalent to some 2nd on $\mathcal{H}_a \otimes \mathcal{H}_b$. The theorem already proved by Tsirelson is that for finite dimensional Hilbert spaces the two situations indeed give rise to the same sets of correlation functions leaving open, up to recently, the infinite dimensional case. The affirmative solution of $\neg$TC on infinite dimensional Hilbert spaces has been given in ref. Zhengfeng Ji et al. 2022 by employing in particular model theory and Turing uncomputability classes the tools we are also using in this work. We think the tools are not accidental but essential for the QM formalism leading to a new understanding of QM.

Given an infinite dimensional separable complex Hilbert space $\mathcal{H}$ there always exist an isometric isomorphism between it and the Hilbert space ℓ_2 of square-summable infinite sequences of complex numbers. Let $V : \ell_2 \to \mathcal{H}$ be such an isometry and $V^* : \mathcal{H} \to \ell_2$ its adjoint. Following ref. Zhengfeng Ji et al. 2022 let the simplified measurement be given by projective valued measures (PVM) i.e. by the collection of projections $\{P_i^k\}, i = 1, 2 \cdots, m$, $\sum_{i=1}^m P_i^k = \mathbb{1}$ and there are n such POVs, i.e. $k = 1, 2 \cdots, n$ for a. Similarly there are n many PVMs for b observer i.e. $\{Q_j^l\}, j = 1, 2 \cdots, m$, $\sum_{j=1}^m Q_j^k = \mathbb{1}$ and $l = 1, 2 \cdots, n$. There results the set of correlations (see (5) and (6) in the Results section for the explicit use of states) between these POVs of a and b on $\mathcal{H}$ (assuming the commutativity $[P_i^k, Q_j^l] = 0, i, j = 1, 2, \cdots, m; k, l = 1, 2, \cdots, n$):

$$\{Corr(\mathcal{A} \cdot \mathcal{B})_{ij}^{kl}\} = \{V^* P_i^k \cdot Q_j^k\ V\}.$$

On the product space $\mathcal{H}_a \otimes \mathcal{H}_b$ an isometry V reads now $V : \ell_2 \to \mathcal{H}_a \otimes \mathcal{H}_b$ with $V^* : \mathcal{H}_a \otimes \mathcal{H}_b \to \ell_2$ and the PVMs of a are now $P_i^k \otimes \mathbb{1}_b$ and PVMs of b $\mathbb{1}_a \otimes Q_j^k$ $i, j = 1, 2, \cdots, m; k, l = 1, 2, \cdots, n$. The corresponding set of correlations now reads

$$\{Corr(\mathcal{A} \otimes \mathcal{B})_{ij}^{kl}\} = \{V^* P_i^k \otimes Q_j^k\ V\}.$$

Then the Tsirelson's conjecture is the statement that for any such settings and any $m, n \in \mathbb{N} \setminus \{0\}$ the sets of correlations $Comp(\{Corr(\mathcal{A} \otimes \mathcal{B})_{ij}^{kl}\})$ and $\{Corr(\mathcal{A} \cdot \mathcal{B})_{ij}^{kl}\}$ are the same, where $Comp(A)$ is the topological completion of the set A.

The measurements in QM of a quantum system S are formally represented by more general positive operator valued measures (POVM) which extends projection valued measures assigned to pure states, such that POVM can comprise also mixed states ρ.

3. Results

Let us take a closer look at the Landsman's equivalence from the Introduction from the point of view of generalised products from different Turing classes. One can stratify directly the equivalence in a way which does not affect it: given the equivalence of A and B, the reduction of A to B is not necessary any r.e. process, so that the higher Turing degrees of the products of Hilbert spaces are allowed.

A'. s is generated in a measurement performed on the 'whole run system' with $\mathcal{H}^{(\infty)} = \bigotimes_{s_i}^{\infty} \mathcal{H}_{s_i}^{(2)}$, however, $s = \{s_i, i \in \mathbb{N}\}$ is not r.e. which means it is in a higher Turing class. Then the statistical results on the ensemble of ss determine the probability measure P^{∞} on $\mathcal{H}^{(\infty)}$. One considers the 'whole run system' as a quantum system on $\mathcal{H}^{(\infty)}$ on which there are performed measurements.

A. s is generated in a measurement performed on the 'whole run system' with $\mathcal{H}^{(\infty)} = \bigotimes_{i=1}^{\infty} \mathcal{H}_i^{(2)}$ with r.e. sequence of Hilbert spaces η and then, the statistical results on the ensemble of ss determine the probability measure P^{∞} on $\mathcal{H}^{(\infty)}$.

B. s is retrieved by collecting the statistical results at each ith, i.e. performed on $\mathcal{H}_i^{(2)}$, and thus concluding about the limiting statistical probability of the sequences in 2^{ω}.

A' is not Turing reducible to B since the preparing of states procedures in B are understood as given by r.e. procedures. Thus more precisely, when A' contains non r.e. sequences it might be nonequivalent to B. but also nonequivalent to A., if one assumes the equivalence of A. and B.. So far we have a direct relativisation to different Turing classes which follows the obvious distinctions

$$(\mathcal{H}^{(\infty)}, \eta) \not\simeq (\mathcal{H}^{(\infty)}, \eta') \text{ i.e. } \mathbf{a}_{\sigma(\eta)} \not\simeq \mathbf{a}_{\sigma'(\eta')} \text{ and } \mathcal{H}^{(\infty)} = \bigotimes \eta, \mathcal{H}^{(\infty)'} = \bigotimes \eta'.$$

Here the sequence $\sigma(\eta)$ is the binary infinite $(0, 1)$ sequence of indices of the sequence η of finite dimensional Hilbert spaces.

Still the consequence regarding the 'absolute' irreducibility of A' and B is not decided and needs a justification. As we have noted already in the Introduction the procedure of hyper-Turing tensoring the sequences of Hilbert spaces leads as a rule to presumably a nonseparable infinite dimensional space as does any ITP of the Hilbert spaces of dimension at least 2. Still it holds

Proposition 1 *Allowing for generalised Turing uncomputable products of Hilbert spaces the collecting statistical data on the whole-run-system as in A' is equivalent to the step-by-step collecting statistical data as in B which is equivalent to A.*

This proposition follows from the Landsmann's result supporting the statistical equivalence of ITP in A with the single turns performed on the ith Hilbert space in the product and from the von Neumann's result showing the invariance of ITP to different arbitrary orderings of the countable infinite sets of indices of the spaces in ITP $\mathcal{H}^{(\infty)}$Von Neumann 1939.

Remaining in the countable infinite number of factors in ITP of Hilbert spaces can we testify the limits of the equivalence A and B in QM? This is what we want to explore now by defining the *ZFC-twisted ITP* of Hilbert spaces.

Remark 2 *Turning to ZFC is based on two important observations. One follows from Theorem 3 from the Key terminologies section stating that the most general perspective on uncomputable sequences requires random sequences which are related with random forcing. The other observation is that the generalisation of the arithmetic forcings reduced to PA is the extension to ZFC random and Cohen forcings. Thus we fix a general perspective as this assigned to ZFC.*

So we refer to ZFC random sets of indices in ITPs. The question arises: are we still within the range of QM? In ref. Król, Bielas, and Asselmeyer-Maluga 2023 it was shown that QM on infinite dimensional Hilbert spaces is indeed set theory Solovay generic, i.e. ZFC-random. This means that certain binary sequences of QM outcomes were formally described as 'generic random real numbers' known from set theory. This requires, however, certain additional suppositions. So thus following ref. Król, Bielas, and Asselmeyer-Maluga 2023 let us assume the working criterion for ZFC randomness of QM:

> [ZFC random] QM on $\mathcal{H}^{(\infty)}$ is ZFC random when there exists a binary random Solovay generic sequence of outcomes, represented by $r \in \mathbb{R}$, generated by the maximal complete atomless Boolean algebra of projections B in the lattice of projections $\mathbb{L}(\mathcal{H}^{(\infty)})$. To be generated by the algebra B means that r is generic over B in V or, perhaps, in some other transitive, standard model M of ZFC.

We refer the reader to Subsection 4.5 for a more detail analysis of the ZFC twist of QM from the point of view of the standard postulates of QM.

Remark 3 *B becomes internal algebra in V (M) and thus the randomness of $\sigma \in 2^{\omega}$ is also reduced to V or M, where 2^{ω} is now $2^{\omega}_{|V}$ or $2^{\omega}_{|M}$.*

We want to understand the randomness of $\mathbb{R} \ni r = \sigma \in 2^{\omega}$ for the Tsilerson's conjecture by allowing for two stages of randomness for infinite sequences of QM outcomes: one is the randomness in the product of $\mathbb{C}^2$ Hilbert spaces, where at each ith entry there is a r_i outcome resulting from the measurement procedure performed on $\mathbb{C}_i$. The other stage of randomness is given by the set of indices $\{i_k\}_{k\in\mathbb{N}}$ which is the binary sequence $\sigma \in 2^{\omega}$ which can be Solovay random (relative to a model M).

Remark 4 *Given an infinite set of real numbers $\{r_{i_k}, k \in \mathbb{N},\}$, with Solovay random indices $\{i_k\}_{k\in\mathbb{N}}$ the set $\{r_{i_k}\}$ is a random sequence of real numbers.*

It follows that we do not need to refer to any quantum process for generating the real entries of the sequence $\{r_{i_k}, k \in \mathbb{N},\}$ and can focus on purely random or uncomputable properties of its index set. Thus let us introduce the ZFC randomness into the strata of ITPs. The procedure relies on the ITPs $\mathcal{H}^{(\infty)} = \bigotimes \eta(\mathbb{C}^2)$, however, the binary sequence $\sigma(\eta) \in 2^{\omega}$ of indices is now considered *ZFC random*. This generalises (or twists) the arithmetic ML n-random sequences. ML n-randomness requires just passing the nth ML test, e.g. ref. Downey and Hirschfeldt 2010, which is fulfilling certain arithmetical properties by $\sigma(\eta) \in 2^{\omega}$. This does not refer to any model of ZFC, similarly the C-genericity and arithmetic Solovay genericity do not refer to such models. The next step, however, requires performing the *ZFC twist* as in Remark 2 and choosing a ZFC model extending arithmetic approach appropriately. This is a kind of broader formal perspective (models of ZFC) which is to be included into the analysis, since the perspective of axiomatised formal PA does not suffice in particular for addressing QM as ZFC random.

Proposition 2 *Let $\sigma = \{i_k\}_{k\in\mathbb{N}} \in 2^{\omega}$ be a ZFC random binary sequence of indices relative to a transitive standard ZFC model M. Then $\bigotimes_{i_k}^{\infty} \mathbb{C}^2_{i_k}$ determines the ITP $\mathcal{H}^{(\infty)}$ and certain s.a. operator $\mathcal{A}$ on a separable infinite dimensional Hilbert space $\mathcal{H}^{\infty}$.*

For the proof see supplementary subsection 4.1.

Corollary 1 *Let $\{a_{i_k}\}, k \in \mathbb{N}$ be a sequence of s.a. commuting operators on $\mathbb{C}^n$ and $\{i_k\}_{k\in\mathbb{N}}$ be a ZFC random binary sequence of indices. Then taking the tensor product $a_{i_1} \otimes a_{i_2} \otimes \cdots$ gives rise to $a \oplus \mathcal{A}$ where a is a s.a. operator on $\mathcal{H}^{(\infty)}$ and $\mathcal{A}$ is s.a. operator on $\mathcal{H}^{\infty}$.*

We will see in what follows that the appearance of operators like $\mathcal{A}$ above may help distinguishing TC from $\neg$TC on $\mathcal{H}^{\infty}$. We are proving the following result

Theorem 4 *If QM is ZFC random then the Tsirelson's conjecture fails.*

To show this let us start with general remarks. ZFC randomness of QM does not mean that all random sequences generated by quantum measurements lead to ZFC random sequences. Instead this is rather an option realised in some conditions. It follows that there might be infinite ZFC random sequences of outcomes as well ML n-random ones or, in general, some other certified differently. But we can always refer to the option and say 'let σ be a ZFC random sequence of QM outcomes on $\mathcal{H}^{\infty}$' and then analyse the conclusions of the existence of such σ.

Thus as defined in [ZFC rand] QM supports the existence of a Solovay generic random binary sequence $r \in \mathbb{R}$ of QM outcomes. This requirement has already strong formal implications. On the one hand the existence of random genericity in QM refers to models of ZFC, which has been partly recognised in ref. Król, Bielas, and Asselmeyer-Maluga 2023. On the other hand this is a general question regarding the existence of generic reals in models of set theory and this fixes the perspective into more formal. One proves in ZFC that the generic filters G for the algebra B in M exist for a CTM M so thus Solovay generic reals added by random forcing to M (note that B - the maximal Boolean algebra of projections in $\mathbb{L}(H^{\infty})$ is an *atomless* complete measure algebra). For Boolean-valued models, M^B, of ZFC such generic reals r exist with Boolean value 1 (provided B being atomless in M). Moreover, the 2-valued model of ZFC (the forcing extension of M) containing the reals r, $M[G]$, are obtained as the quotient model, i.e. $M^B/G = M[G]$. For the case of universe V of set theory one defines the Boolean valued model V^B in V (but still the canonical embedding $V \hookrightarrow V^B$ exists) and generic reals do exist with the Boolean value 1 (see the supplementary subsection 4.1). Such approach is also the resolving of the issue of nonexistence of generic filters in V such that, from the point of view of the cumulative universe of sets V considering generic random sequences requires referring to a CTM M or to Boolean-valued models like V^B.

Remark 5 *There are various results in set theory which enable genericity, e.g. allowing for the negation of the continuum hypothesis (CH) it is possible (let $\mathfrak{c}$ represents continuum) that there are cardinalities κ, $\aleph_0 < \kappa < \mathfrak{c}$, for which generic filters exist and there are Martin's axioms MA(κ) stating this (MA(κ) are independent of ZFC and consistent with $\neg$CH). There exist also the refined versions of forcings fulfilling variety of forcing axioms, like proper forcing, enabling the genericity. At this stage we do not think that the refined results as above should be addressed in the context of QM currently.*

The proof goes from this standpoint of the universe V.

Thus let us turn to a CTM M which supports the existence of r generic over B in M, which guarantees that QM be ZFC random. We are building the binary sequences of indices for which there correspond s.a. operators on $\mathcal{H}$ which additionally may (or may not) be correlated with the operator $\mathcal{A} = \mathcal{A}_r$ from the Proposition 2.

The fundamental property of generic extensions related to a CTM M says that (e.g. ref.Bartoszyński and Judah 1995)

Lemma 1 *For any random generic $r \in M[r]$ there does not exist any Cohen generic $c \in M[r]$ such that $r = c$. For any Cohen generic $c \in M[c]$ there does not exist any Solovay generic $r \in M[c]$ such that $c = r$.*

Remark 6 *'Generic' here means random or Cohen reals not in the ground model M but in $M[r]$ or $M[c]$ correspondingly and such generic reals correspond to the generic filters in the complete random, B, and Cohen, C, Boolean algebras in M. Reals in a model M are those binary sequences which are in 2^{ω} in M, i.e. 2^{ω}_M. Thus the lemma states that among random generic reals in the extension there are no Cohen generic, and conversely.*

We have two results

Lemma 2 *In V^B real numbers are in 1:1 correspondence with s.a. commuting operators in B on $\mathcal{H}^{\infty}$.*

Lemma 3 *Let c be a Cohen and r a random generic reals, $R_{M[c]}$ the reals in $M[c]$ and $R_{M[r]}$ the reals in $M[r]$. Then it holds: $c \in R_{M[c]} \subset \mathbb{R}$ and $r \in R_{M[r]} \subset \mathbb{R}$ and $\mathbb{R} \hookrightarrow R_{V^B}$.*

Lemma 2 is the direct conclusion from Lemmas 1 and 2 in the SM A file. Lemma 3 above recapitulates the elementary relations for CTMs and Boolean models of ZFC Jech 2003; Bell 2005.

So let M be the standard transitive ZFC model (possible countable) according to which $r = \{i_k\}_{k\in\mathbb{N}}$ is generic random and $c = \{j_k\}_{j\in\mathbb{N}}$ be a binary sequence which is Cohen generic again with respect to M. Even though now we have the Cohen complete Boolean algebra C in M with respect to which $c = \{j_k\}_{j\in\mathbb{N}}$ is generic and C is not chosen from the lattice of projections and the indices are not connected with the operators and hence the lattice of projections. Still one can formulate the analogous result like for the measure algebra B

Corollary 2 *Let $\{a_{j_k}\}, k \in \mathbb{N}$ be a sequence of s.a. commuting operators on $\mathbb{C}^n$ and $\{j_k\}_{k\in\mathbb{N}}$ be a ZFC Cohen generic binary sequence of indices with respect to M. Then the tensor product $a_{j_1} \otimes a_{j_2} \otimes \cdots$ determine $a \oplus \mathcal{A}_c$ where a is a s.a. operator on $\mathcal{H}^{(\infty)}$ and $\mathcal{A}_c$ is certain s.a. operator on $\mathcal{H}^{\infty}$.*

This follows from Lemma 3 since $c = \{j_k\}_{j\in\mathbb{N}} \in 2^{\omega}_M \subset \mathbb{R}$ and $\mathbb{R} \hookrightarrow R_{V^B}$ and from Lemma 2 so one is choosing $\mathcal{A}_c$ as corresponding to $c \in R_{V^B}$. The first part of this Corollary refers to the of s.a. operator a which is the ITP limit of the corresponding finite tensor products completed (by the products of identities) (see ref. Landsman 2020).

From Corrolaries 1 and 2 it follows

Proposition 3 *There exist s.a. operators $\mathcal{A}^M_c, \mathcal{A}^M_r$ on $\mathcal{H}^{\infty}$, corresponding to Cohen and random generic reals with respect to any CTM M of ZFC.*

We are going to recognise properties of the sets of correlations $Corr(\mathcal{A}^M_r, \mathcal{A}^M_c)$ of the operators $\mathcal{A}^M_c, \mathcal{A}^M_r$ on $\mathcal{H}^{\infty}$ vs. on products $\mathcal{H}_a \otimes \mathcal{H}_b$ of Hilbert spaces. It holds

Theorem 5 *Let M be a CTM of ZFC, then*

a) $[\mathcal{A}^M_r, \mathcal{A}^M_c] = 0$.

b) There exist certain correlations $Corr(\mathcal{A}^M_r, \mathcal{A}^M_c)$ on $\mathcal{H}^{\infty}$ which can not be reproduced by any correlations $Corr(a \otimes \mathbb{1}, \mathbb{1} \otimes b)$ on $\mathcal{H}_a \otimes \mathcal{H}_b$ with a, b s.a. on any $\mathcal{H}_a$ and $\mathcal{H}_b$ correspondingly.

The proof goes as follows. First let us check the commutativity of $\mathcal{A}^M_r, \mathcal{A}^M_c$ in V. The random real $r \in R_{M[r]}$ and the Cohen real $c \in R_{M[c]}$ are also reals in $\mathbb{R}$ in V (Lemma 3). Then from Lemma 2 it follows that commuting s.a. operators on $\mathcal{H}^{\infty}$ in B in V are in 1:1 correspondence with reals R_{V^B} in V^B (see the SM A file) and $\mathbb{R} \hookrightarrow R_{V^B}$. From the constructions of both operators (Corollaries 1 and 2) it follows that they correspond to $r, c \in R_{V^B}$ so $[\mathcal{A}^M_r, \mathcal{A}^M_c] = 0$.

Next let us consider the set of correlations $Corr(\mathcal{A}^M_r, \mathcal{A}^M_c)$ on $\mathcal{H}^{\infty}$. It holds (relative to M)

$$\text{Given } \mathcal{A}^M_r \text{ the operator } \mathcal{A}^M_c \text{ is excluded;} \tag{1}$$

$$\text{Given } \mathcal{A}^M_c \text{ the operator } \mathcal{A}^M_r \text{ is excluded.} \tag{2}$$

This is due to Lemma 1 stating that given a random real r in $M[r]$ there can not exist any generic Cohen real c in $M[r]$ and conversely, so thus the operators corresponding to the generic reals inherit this property.

Let us consider now the set of correlations $Corr(a \otimes \mathbb{1}, \mathbb{1} \otimes b)$ on $\mathcal{H}_a \otimes \mathcal{H}_b$. The strong negative correlations as in (1) and (2) are not reproducible on the product $\mathcal{H}_a \otimes \mathcal{H}_b$. To see this let $c \in M[c]$ be Cohen generic real and $r \in M[r]$ a random generic real and $\mathcal{A}^M_r$, $\mathcal{A}^M_c$ be the corresponding s.a. operators on $\mathcal{H}_a$ and $\mathcal{H}_b$ correspondingly. Let us suppose that $\mathcal{H}_a = \mathcal{H}^{\infty}_a$ and $\mathcal{H}_b = \mathcal{H}^{\infty}_b$. This supposition is not any limitation since taking two finite dimensional Hilbert spaces their product already can not reproduce all correlations on $\mathcal{H}^{\infty}$. To show that the exclusions (1), (2) do not hold on the product $\mathcal{H}_a \otimes \mathcal{H}_b$ let us take $\mathcal{A}_r^M \otimes \mathbb{1}$ and $\mathbb{1} \otimes \mathcal{A}_c^M$ on $\mathcal{H}_a \otimes \mathcal{H}_b$. One sees that

$$\text{Given } \mathcal{A}_r^M \otimes \mathbb{1} \text{ the operator } \mathbb{1} \otimes \mathcal{A}_c^M \text{ is not excluded;} \tag{3}$$

$$\text{Given } \mathcal{A}_c^M \otimes \mathbb{1} \text{ the operator } \mathbb{1} \otimes \mathcal{A}_r^M \text{ is not excluded.} \tag{4}$$

The reason for this non-exclusions is that now we have independence of Hilbert spaces in the product which extends over independence of generic extensions and the corresponding operators which means (in the first case of $\mathcal{H}^\infty$) $M[r]$ excludes $M[c]$ and conversely, while in the case of $\mathcal{H}_a \otimes \mathcal{H}_b$ the extensions appear freely. This finishes the prove of Theorem 5.

To complete the proof of Theorem 4 one can not deal directly with the $\mathcal{A}_r$ and $\mathcal{A}_c$ operators and their correlations since they are presumably not finitely decomposable like $\mathcal{A} = \sum_{i=1}^N \lambda_i P_i, \lambda_i \in \mathbb{R}, i = 1, 2, \cdots, N$. However, the existence of $\mathcal{A}_r$ and $\mathcal{A}_c$ indicates on the existence of the extended universes $M[r]$ and $M[c]$ in which one can find generic finitely decomposable operators $\mathcal{A}^{M[r]} = \sum_{i=1}^N \lambda_i{}^{(r)} P_i{}^{(r)}, \lambda_i{}^{(r)} \in R_{M[r]}, i = 1, 2, \cdots, N$ for its finitely many eigenvalues and spaces in $\mathcal{H}^{M[r]}$ in the extended model $M[r]$ such that some of its eigenvalues are M B-generic. Similarly for certain s.a. operator $\mathcal{A}^{M[r]}$ in $M[c]$ such that $\mathcal{A}^{M[c]} = \sum_{i=1}^N \lambda_i{}^{(rc} P_i{}^{(c)}, \lambda_i{}^{(c)} \in R_{M[c]}, i = 1, 2, \cdots, N$ which some of its eigenvalues are M C-generic. It should also hold $\sum_{i=1}^N P_i{}^{(r)} = Id$ on $\mathcal{H}^\infty$ in $M[r]$ and $\sum_{i=1}^N P_i{}^{(c)} = Id$ on $\mathcal{H}^\infty$ in $M[c]$ and $[P_i{}^{(r)}, P_j{}^{(c)}] = 0$ for all $i, j = 1, 2, \cdots, N$. How do we know such generics exist? Let us start with certain s.a. operators $\mathcal{P}$, $\mathcal{Q}$ on $\mathcal{H}^\infty$ in V with the properties we need

$$\mathcal{P} = \sum_{i=1}^N \lambda_i P_i, \lambda_i \in \mathbb{R}, i = 1, 2, \cdots, N$$

$$\mathcal{Q} = \sum_{i=1}^N \kappa_i Q_i, \kappa_i \in \mathbb{R}, i = 1, 2, \cdots, N$$

$$\sum_{i=1}^N P_i = \sum_{i=1}^N Q_i = Id \text{ on } \mathcal{H}^\infty \text{ and } [P_i, Q_j] = 0 \text{ for all } i, j = 1, 2, \cdots, N \text{ (in } V) .$$

Projections P_i, Q_j as well $\mathcal{H}^\infty$ can be defined in $M[r]$ and $M[c]$ Benioff 1976a. For example given the projection say

$$P_i = \text{ 'the projection on } i\text{th 1-dimensional complex subspace in } \mathcal{H}$$

and this can be directly interpreted in $M, M[c], M[r]$ just remembering that the complex numbers describing the subspace are pairs of reals from $R_M, R_{M[r]}, R_{M[c]}$ the subsets of $\mathbb{R}$. Real numbers (eigenvalues) $\lambda_i, \kappa_j, i, j = 1, 2 \cdots, N$ are taken in such a way that at least one λ_i (at least one κ_j) are M B-generic in $M[r]$ (M C-generic in $M[c]$). This choice requires eventual tiny modifications of real coefficients since for the Cohen generic extension $M[r]$ the generic reals have Lebesgue measure 1 (full) in $R_{M[c]}$ and the random generic have outer measure 1 in $R_{M[r]}$. Now let us take $\mathcal{A}^{M[r]} = \mathcal{P}$ and $\mathcal{A}^{M[c]} = \mathcal{Q}$ which have the desired properties.

Note that working in V we do not face the obstructions like in (1), (2) since the nontrivial generics do not exist in V. The central question seems to be: What is the QM reason that experiments alignments and QM are shrunk and collapse to the models like $M[r], M[c]$? The answer is *genericity* of QM on $\mathcal{H}^\infty$ (see Subsections 4.2 and 4.5 and Lemma 6). What is more, when observables $\mathcal{P}$ and $\mathcal{Q}$ commute in V (i.e. the corresponding projections commute) they are simultaneously measured.

Thus given 'QM is ZFC random' there exists a random generic $r \in M[r]$ of outcomes of QM and QM is in $M[r]$. We will show that the strong negative correlation (1) vs. uncorrelated case (3) apply for $\mathcal{A}^{M[r]} = \mathcal{P}$ in $M[r]$ and $\mathcal{A}^{M[c]} = \mathcal{Q}$ in $M[c]$ as above.

Let us describe the usual physical settings for TC N. Ozawa 2013. On $\mathcal{H}^{\infty}$ Alice (a) is performing measurements over the system S and she has a finite set of positive bounded operators $\mathcal{A}_{i\alpha} \in \mathbb{A}_{\{i,\alpha\}} \subset \mathcal{L}(\mathcal{H}^{\infty})$ where i is the index of the ith operator and α indicates certain outcome resulting in the measurement of operators from $\mathbb{A}_{\{i,\alpha\}}$. There are finitely many of the all possible outcomes αs. Similarly Bob (b) is performing independent measurements over S with $\mathcal{H}^{\infty}$ and he has a finite $\mathbb{B}_{\{j,\beta\}}$ set of positive bounded operators indexed by j with the finite set of all possible outcomes βs. Under the assumption that all $\mathcal{A}_{i\alpha} \in \mathbb{A}_{\{i,\alpha\}}$ commute with all $\mathcal{B}_{j\beta} \in \mathbb{B}_{\{j,\beta\}}$ (independence of measurements by a and b) and given a state ω – a positive linear normalised functional $\omega : \mathcal{L}(\mathcal{H}^{\infty}) \to \mathbb{R}$, one can represent the set of possible correlations arising in the measurements of S by a and b on $\mathcal{H}^{\infty}$

$$Corr(\mathcal{A} \cdot \mathcal{B}) = \{prob(i,j|\alpha,\beta) = \omega(\mathcal{A}_{i\alpha} \cdot \mathcal{B}_{j\beta}) \text{ on } \mathcal{H}^{\infty}\}_{ij\alpha\beta}. \tag{5}$$

On $\mathcal{H}_a \otimes \mathcal{H}_b$ the independence of measurements by a and b is guaranteed by the separation between the corresponding Hilbert spaces of S on which a and b are performing their measurements. The state ω is now the functional $\omega : \mathcal{L}(\mathcal{H}_a \otimes \mathcal{H}_b) \to \mathbb{R}$, $\mathbb{A}_{\{i,\alpha\}} \subset \mathcal{L}(\mathcal{H}_a)$, $\mathbb{B}_{\{j,\beta\}} \subset \mathcal{L}(\mathcal{H}_b)$ which results in the set of correlations

$$Corr(\mathcal{A} \otimes \mathcal{B}) = \{prob(i,j|\alpha,\beta) = \omega(\mathcal{A}_{i\alpha} \otimes \mathcal{B}_{j\beta}) \text{ on } \mathcal{H}_a \otimes \mathcal{H}_b\}_{ij\alpha\beta}. \tag{6}$$

The TC is the hypothesis that for any settings as above $Corr(\mathcal{A} \cdot \mathcal{B}) = Corr(\mathcal{A} \otimes \mathcal{B})$. Let us apply the generic QM into such formulated TC. Working in V both a and b are performing their experiments in the settings above and both B-genericity (QM is ZFC random) and C-genericity are available. In the measurement on $\mathcal{H}^{\infty}$ the set universe V (context) is collapsed, $V \to M$, such that the random extension after the measurement $M[r]$ is nontrivial (see SM B). Then according to Lemma 6 the reduced operator $\mathcal{A} \to \mathcal{A}^{M[r]}$ is measured. Similarly to Bob's measurement $\mathcal{B}$ is collapsed to $\mathcal{B}^{M[c]}$ and measured in $M[c]$. Thus there is the possibility that $\mathcal{A} \in \mathbb{A}_{\{i,\alpha\}}$ and the set of αs includes eigenvalues of $\mathcal{A}$ (finite number) and $\mathcal{B} \in \mathbb{B}_{\{j,\beta\}}$ and the eigenvalues of $\mathcal{B}$ are in the set of βs. Then let $\lambda_i \in R_{M[r]}$ be one of the generic eigenvalues of $\mathcal{A}^{M[r]}$ in $M[r]$ and κ_j be generic of $\mathcal{B} = \mathcal{A}^{M[c]}$ in $M[c]$. Then the strong negative correlation of (1) applies to $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{H}^{\infty}$ for the projections corresponding to these generic eigenvalues.

Still with $\mathcal{A} = \mathcal{A}^{M[r]}$ and $\mathcal{B} = \mathcal{A}^{M[c]}$ let the Hilbert space be the product $\mathcal{H}_a \otimes \mathcal{H}_b$ and the operators result as $\mathcal{A}_a^{M[r]} \otimes \mathbb{1}$ and $\mathbb{1} \otimes A_b^{M[c]}$. They do not face the negative correlations above and hence follow the pattern of (3) and (4) and, since there are no obstructions coming from single models and generics the negative correlations do not emerge and are not reproducible here. This completes the proof of Theorem 4.

Remark 7 *One could wonder whether the asymmetry in assigning the random extensions to a and Cohen extension to b is valid. In fact it can be performed conversely; it is also possible that* $M[r]$ *is assigned to both a and b or that* $M[c]$ *is assigned to the both and this does not change the result. In fact one should follow the rule of equal accessibility of the choices by both participants. Otherwise there would be dependence of choice b on the choice a which we want to exclude. Still the choice in the proof is possible and when followed leads to the result; see SM C file for this generalised standpoint.*

Remark 8 *One could also wonder whether genericity is indeed required for the result of Theorem 4. Let us assume that c and r were non-generic and they exist already in M as reals from* R_M *so they still exist in* V^B *and correspond to commuting s.a. operators. But then the exclusions* (1) *and* (2) *can not hold since Lemma 1 does not hold for non-generic reals.*

Thus in this approach the form of amplified randomness of binary sequences related with the ZFC genericity is indeed needed for negating the Tsirelson's conjecture.

The presented above results touch a fundamental issue in set theory - the existence of generic ultrafilters in general, however, it seems that the issue is somehow decisive in understanding randomness in QM on infinite dimensional Hilbert spaces, e.g. ref. Król, Bielas, and Asselmeyer-Maluga 2023. Genericity becomes a part of formalism of QM especially measurement process. This will be more thoroughly discussed in the next section (see also SM E). The QM contexts based on ZFC are models M, $M[G]$ and the like which become the 'worlds of classical discourse', or classical contexts, and come up here as bearing the nontrivial formal structure. The contexts, however, are not only representing the classical world of physics but influence also the quantum picture. This is particularly seen in the non simultaneity of contexts $M[r]$ and $M[c]$ which could be seen as yet another quantum like property. This particular point will be addressed separately.

4. Discussion and Supplementary Material

In this explanatory long section we present deepened discussion of various fundamental, though more distant to Theorem 4, aspects of QM formalism. Nevertheless, the Reader can find also here, as subsection 4.1, the proof of Proposition 2 missing in the main body of the paper.

In mathematics, the search for an answer to the question of whether generic filters (in particular over V) exist puts the problem in the spotlight alongside other ontological questions – and hence, in a sense, outside of the formalism of set theory Hamkins 2012. The existence of generic ultrafilters is not universal among models of ZFC, as was already emphasized earlier (see the SM A file). In particular, they do not exist for V and always exist for a CTM M. One solution is to refer to Boolean models like V^B – then generic ultrafilters will exist with the Boolean value 1. From the forcing point of view, referring to Boolean models in V is closely related to construing CTMs in V. Such an approach fixes V as an absolute environment that, in particular, cannot be extended: for instance, adding new reals to $R_V = \mathbb{R}$ will be pointless, as $\mathbb{R}$ already contains all reals, just as V should contain 'all' sets. However, the absolute character of the universe of sets (cumulative hierarchy of sets) can be relativized in the foundations of set theory, leading to the multiverse-based (MV-based) foundations. One of the main problems leading to MV is the relation to the existence of generic filters and extensions such as $V[G]$. As Hamkins explains, the point Hamkins 2012 generic filters G do not exist in V, but this should be viewed analogously to the nonexistence of $\sqrt{-1}$ in $\mathbb{R}$, and the extensions $V[G]$ would then correspond (only by historical similarity rather than by any actual correspondence) to $\mathbb{C}$. The Forcing Extension Principle, which is one of the axioms of the MV approach, states that *for any universe V and any forcing notion P in V, there will be a forcing extension $V[G]$, where $G \subset P$ is V-generic.* What strikes us as strange is that there seems to be yet another principle of MV in this direction, i.e. Absorption into **L** (the constructible Gödel universe), where this says that every universe V will be a countable transitive model in another universe W that satisfies $V = \mathbf{L}$. Not only is V in W countable, but also W can be chosen as a constructible universe. However, a usual feature of this approach is that V is typically a non-well-founded (nonstandard) model as seen by W. Thus, MV places us in a realm where countability and well-foundedness are mutually relative aspects, and there will not be any absolute or distinguished universe V fixing the meaning of countability and well-foundedness. These two radically different approaches in the foundations of set theory, one based on universe V and the other on multiverse MV, now belong to the scope of QM considerations.

We seem to be at a rather peculiar point, where experiments can shed light on an abstract but crucial decision in the foundations of set theory: namely, whether or not generic filters in V exist, and whether the V-based or MV-based approach is more likely to prevail over the other one. The approach to TC presented here assumes the ZFC genericity of QM and the existence of ZFC random sequences of outcomes, but nevertheless it may happen to be verified experimentally (along with the distinction TC and $\neg$TC). In Subsection 4.5, we have shown that ZFC genericity is a part of the standard QM formalism when completed by the set theory part.

Three scenarios are therefore possible: the first points to M and its generic extension $M[G]$

in which QM is formulated and M would be physically real - i.e. with V excluded due to the $\neg$(V-genericity) property. This would mean that set theory, construed as underlying our physical reality, is shrunk down to model M, but universe V is still a suitable point of reference for describing M, and V-based theoretical foundations would thus prevail over MV ones. In classical physics, or for non-extreme energy scales, this distinction could prove irrelevant. If it happens that experiments really show that QM is ZFC generic, then the V-based formalism of classical physics will not be extendable over QM, which would have to be formulated in a ZFC model M supporting the existence of generic filters – or, alternatively, the paradigm would have to be radically changed. This second paradigm points to the MV-based foundations of set theory, where genericity is assured at every stage, where this would dominate in the quantum regime (on $\mathcal{H}^{\infty}$).

There is also a third possibility, which seems most natural from the point of view of the methods adopted in this paper: namely, that true universes are local Boolean-valued models V^B. This would not contradict the V-based approach, realizes genericity, and is closed to the CTMs, as well as following directly from the QM formalism. (See the discussion in the Subsection 4.5). Moreover, this last option seems to be well-suited for addressing an issue such as the overlapping domain of space-time with a quantum regime – something that will be a topic of a separate work from the present authors. (However, see ref. Król and Asselmeyer-Maluga 2020).

There is a certain formal context which already favors the MV approach in QM and is related to hidden variables of QM. This is the mutual genericity of QM. The appearance of genericity in QM has already been analysed by Robert van Wesep (Van Wesep 2006), William Boos (Boos 1996) and Paul Benioff (Benioff 1976a, 1976b), and their arguments referred to in ref. Król, Bielas, and Asselmeyer-Maluga 2023 (as well as in a series of papers by the present authors and their collaborators – see, e.g., Król and Asselmeyer-Maluga 2022; Król 2004, 2016; Król and Klimasara 2020). In the context of quantum physics there is also a work by Illias Farah and Menachem Magidor (Farah and Magidor 2012) showing the independence of Pitovsky's construction of spin chains on ZFC axioms. The method applied is forcing, and some large cardinal supposition. Still, not many papers so far have focused on the issue of genericity in QM. The message coming from the works by Wesep and Boos is that if Solovay genericity were to be removed from the QM formalism, this would contradict the nonexistence of hidden variables in their strong form in QM. That means, in particular, that QM is formulated in a formal ZFC environment which supports mutual genericity: if it were V at some stage then, formally, there would not be any generic filter, and hence no genericity would be possible. It should be MV or V^Bs, as $V^B \subset V$ reintroduces genericity in V. (See the supplementary subsection 4.5.) Genericity has also been present in QM – albeit in an implicit, finite form – in a work by Paterek et al. 2010. The authors in question have shown that logical independence, and hence forcing-related construction, is a strong principle in QM, such that it leads to quantum phenomena. This work by Paterek et al. has dealt with finite-dimensional Hilbert spaces, but based on our present work we are able to discern two main directions for extending such an approach. One is the logical independence of finite axiomatic systems over infinite axiomatic ones. The important observation of ref. Paterek et al. 2010 is the replacement, for the Heisenberg group of observables, of quantum randomness by purely classical logical independence. The partial (though maximal) information contained in finite sets of axioms/states concerning experimentally verifiable formulas/sentences is responsible for the inherent randomness of quantum outcomes. This goes in a reverse direction from the usual way of seeing such things. In refs. Król and Asselmeyer-Maluga 2020; Król, Bielas, and Asselmeyer-Maluga 2023, randomness in QM for infinite-dimensional $\mathcal{H}$s has been represented by forcing genericity in models of ZFC, where this points to logical independence on ZFC axioms. ZFC cannot be finitely axiomatized, so such a ZFC-based approach in QM indicates that the infinite-dimensional $\mathcal{H}$ should be related to independence in formal systems having an infinite number of axioms. This, and the finite case of independent sentences, point to the peculiar fact that logical independence may underlie randomness in QM. The other extension of the work in ref. Paterek et al. 2010 comes from quantum-resource theory. (The present authors wish to thank Roberto Salazar for drawing their attention to QRT.) In ref. Król, Bielas, and Asselmeyer-Maluga 2023 we have encountered the possibility of QM being formulated in a model M of ZFC, rather than in the entire universe V of sets. A pinpointing of a certain QRT for the 'QM in *M*' situation might facilitate further pursuit of an eventual experimental verification. These two scenarios are currently subject to investigation, and will be addressed separately.

We have presented an approach in which genericity and forcing (and hence logical independence on ZFC) may lie at the heart of QM on infinite dimensional Hilbert spaces and can allow us, in principle, to distinguish TC and ¬TC. Still, the important issue is whether one could distinguish TC and ¬TC experimentally, and how to make use of this eventually over-correlated phenomenon in practice. One natural expectation would be that we could design and build a new generation of quantum random-number generators certified by ¬TC correlations. As abstract as this looks now, it must nevertheless await future endeavors such as will allow, in particular, a grasp of ¬TC correlations. Certainly, additional - also theoretical - work is needed. This is related to the Solovay randomness or ZFC randomness of QM, which has been assumed here when proving Theorem 4, but the formalism of QM on infinite-dimensional Hilbert spaces is closely related to this Król, Bielas, and Asselmeyer-Maluga 2023 (see the SM E). A related problem, then, is whether the ZFC randomness of QM outcomes can be turned into real new generators. It seems at present that a way towards a proper understanding of the state of the art of ¬TC correlations could go through their relation to the no-signaling theories exploring extensions of standard QM. Again, work on this is currently in progress.

This confirmation of ¬TC continues to stand in a certain analogy with the other problem raised by Tsirelson: i.e. whether the infinite dimension of a Hilbert space can be seen in a finite number of bipartite correlations Tsirelson 1993. This has been resolved in the affirmative in ref. Coladangelo and Stark 2020, where the present authors have shown that five such correlations already serve to distinguish an infinite dimension of Hilbert spaces from a finite one. Thus, in principle, the infinite dimension represents a special and distinct realm of QM; on the other hand, performing real experiments to confirm it seems currently out of reach. The second problem of Tsirelson discussed here presents some further, probably more complex, refinement of the infinite-dimensional case of QM.

The supposition in Theorem 4 that QM is ZFC-random therefore calls for comment. The 'If' part will not be needed anymore in the case discussed above, where there are experimentally verified ZFC-generic sequences in QM, or in the case where ¬TC requires necessarily (referring to) such generic sequences as are not known at present. So far it has been shown that one reason for ¬TC is the genericity, or ZFC randomness, of QM. We have shown in the Subsection 4.5, that QM can already be viewed as ZFC-random, with some elaboration of the set-theory component of QM that is directly determined by the structure of $\mathbb{L}$. However, whether this possibility is indeed adhered to by quantum phenomena must await future verification. From the purely formal point of view, when the set-theory component is represented, QM can be seen as ZFC-random on infinite-dimensional Hilbert spaces.

The use of ZFC models as structural components of physical theories has already proven to be a valid tool, in the sense that ZFC models serve as *physical degrees of freedom* of a kind. In particular, such an approach helps us to understand the regime in which space-time becomes a quantum object Król and Asselmeyer-Maluga 2022, 2020. On the other hand, formal methods based on models of set theory, even though entirely assigned to the domain of ZFC, might well come to be rediscovered and found to be highly useful right at the heart of QM considerations.

4.1 The proof of Proposition 2

Let $\sigma = \{i_k\}_{k\in\mathbb{N}} \in 2^\omega$ be a ZFC random binary sequence of indices relative to a transitive standard ZFC model M.

First we choose a model M as CTM of ZFC. Let $\mathcal{H}^{(\infty)}$ be a separable infinite dimensional complex Hilbert space and $\mathbb{L}(0, 1, \wedge, \vee) = \mathbb{L}$ be its corresponding lattice of projections. The maximal Boolean algebra of projections $\mathcal{B}$ chosen from $\mathbb{L}$ reads Groote 2005; Kadison and Ringrose 1997

$$\mathcal{B} = B_a \oplus B$$

where B_a is certain atomic Boolean algebra corresponding to the finite dimensional, while B is the complete atomless Boolean algebra which is always present for the infinite dimensional Hilbert spaces (and it is absent for the finite dimensions). Moreover, this B is always isomorphic to the measure algebra $Bor(\mathbb{R})/Null$ of Borel subsets of $\mathbb{R}$ modulo the ideal of the subsets of the Lebesgue measure zero.

Let $B_M = B$ be the measure algebra relative to M, i.e. we use the same symbol B for the measure algebra outside the model M (in V) and in M. In M one defines the Boolean-valued universe of sets M^B, however, being construed in M it allows for the canonical embedding $M \hookrightarrow M^B$. M^B is not any 2-valued universe of sets but it is rather the universe with B-valued logic such that the theorems of 2-valued ZFC are formulas with logical value 1. To retrieve the 2-valued model one performs the quotient construction by a generic filter $\mathcal{U}$ in B (in M). The result is the generic 2-valued extension of the model M

$$M^B/\mathcal{U} \simeq M[\mathcal{U}].$$

This $M[\mathcal{U}]$ is precisely the random forcing extension of M. It is marked as $M[r]$ where r is a random real adjoined to M. Usually there are plenty of such generic random reals which come along with each random real r. The point is that for a CTM M of ZFC there always exists a generic filter $\mathcal{U}$ as above so does a generic real. Moreover, whenever B is a complete atomless in M then there always exists the nontrivial forcing extension $M \subsetneq M[r]$ as above. In general, however, such generic filter may not exist.

Now let us turn to $M = V$ the von Neumanns cumulative universe of sets. Here there emerges the problem with non-existence of any generic filter for B in V. The solution is again referring to the construction of Boolean-valued universe of sets V^B in V. Then one proves Viale, Audrito, and Steila 2019, p.70

$$V^B \vdash [\![\exists_{\mathcal{U}}\mathcal{U} \text{ is a generic filter in } B]\!] = 1 \in B \tag{7}$$

which allows to overcome the difficulty and consider the Boolean universe V^B as necessary step in the construction. Then the usual construction for the random forcing extension is valid and reads (to be sure that $V^B/\mathcal{U}$ is standard, one should refer to ω-ZFC models, i.e. with the standard natural numbers object).

$$V^B/\mathcal{U} \simeq V[r], \text{ where } r \text{ is a random generic real (relative to } B \text{ in } V). \tag{8}$$

One such random real represents the binary sequence $\sigma = \{i_k\}_{k\in\mathbb{N}} \in 2^\omega$ of indices as in the Proposition 2 in the main text. The Remark 4 in Results section leads to the conclusion that given a set of Hilbert spaces, say $\mathbb{C}^2_{i_k}, k \in \mathbb{N}$, it is a random sequence of the spaces. But this gives rise to the splitting of the randomness of the indices from the ITP construction performed on spaces. But the first is a random real r and the ITP is a nonseparable $\mathcal{H}^{(\infty)}$. It remains to show that r determines a s.a. operator on a separable H^∞. This follows from general properties of Boolean valued models of ZFC in the context of the QM lattice of projections $\mathbb{L}(\mathcal{H}^\infty)$. V^B is the model of ZFC which means it has an object of real numbers $R^{(B)}$, however, the logic of V^B is Boolean and not 2-valued and the object of reals is non-trivially bigger than real numbers $\mathbb{R}$ in V. This has been analysed already by Takeuti 1978 while developing the Boolean valued analysis. The equation (7) indicates the status of the generic real r as in equation (8). Namely

$$\exists_{r\in R^{(B)}}[\![r \text{ is } V\text{-generic random real}]\!] = 1 \in B.$$

Thus this is not any real $r \in \mathbb{R}$ added to V but rather a Boolean real $r \in V^B$. Now we can interpret this real as follows.

Lemma 4 (Takeuti 1978) *Real numbers in V^B are in 1:1 correspondence with the partitions of unity E in B.*

Remark 9 *We say that a s.a. operator $\mathcal{A}$ is in B when all the projections from defining it spectral family $\{E_\lambda\}$, are in B.*

Lemma 5 (Takeuti 1978) *The partitions of unity E in B are in 1:1 correspondence with the s.a. operators in B on $\mathcal{H}^\infty$.*

This last Lemma follows from the spectral decomposition of any s.a. operator $\mathcal{A}$, i.e. $\mathcal{A} = \int_{sp(\mathcal{A})} \lambda E_\lambda$, $\forall_\lambda E_\lambda \in B$, where $E = \{E_\lambda\}$ is the spectral family of $\mathcal{A}$ and it is the partition of unity in V^B at the same time.

Taking ultrafilter quotient as in (8) we find that in $V[r]$ the Boolean generic real corresponds to certain s.a. operator $\mathcal{A}$. Thus we conclude

Corollary 3 *In V there exists the s.a. operator $\mathcal{A}$ on $\mathcal{H}^\infty$ corresponding to the V-generic Boolean r in $V[r]$.*

And the statement as in Proposition 2 follows: $\bigotimes_{i_k}^{\infty} \mathbb{C}^2_{i_k}$ *determines the ITP $\mathcal{H}^{(\infty)}$ and certain s.a. operator $\mathcal{A}$ on a separable infinite dimensional Hilbert space $\mathcal{H}^\infty$.*

4.2 Measurements in QM and genericity

We extend the measurement rule in V formulated in Król, Bielas, and Asselmeyer-Maluga 2023, Secs. 2.1.2; 3.4.1. In V one has both B and C algebras so thus one can build the sequences which *become* generic over V^B and V^C the Boolean-valued models of ZFC (recall that both algebras, measure and Cohen, are atomless complete Boolean algebras). In V such sequences can coexist since they are not B nor C V-generic. Hence the elavation $V \to V^B \to V^B/\mathcal{U}$ or $V \to M$ leads directly to B- or C-genericity: $V^B/\mathcal{U} \simeq V[\mathcal{U}]$ or $M \to M[\mathcal{U}]$ (same goes for C).

> [QM Generic Measurement] Generic QM measurements on $\mathcal{H}^\infty$ are paired with the random forcing extensions $V \to V[\mathcal{U}]$ or $M \to M[\mathcal{U}]$. The universe V is thus changed to the relative universe V allowing for the nontrivial forcing extensions $V[\mathcal{U}]$. Equivalently V can be seen as replaced by a CTM M in V allowing for the nontrivial $M[\mathcal{U}]$ as well.

We use the same symbol M for the V which allows for nontrivial forcing extensions (e.g. in MV approach) and for M - a CTM replacing V. So V is the cummulative universe of sets not allowing directly for generic extensions. It follows that

Lemma 6 *In a generic measurement on $\mathcal{H}^\infty$ certain C-sequences in V (non-generic) become C-generic in M.*

This follows from the absoluteness of the Cohen algebra for CTMs and thus the construction of generic filters in V in $M \subset V$ become C-generic relative to M. Given a CTM M there always exist C-generic reals and $R_M \subset R_{M[c]} \subset \mathbb{R}$.

4.3 Generic extensions and ¬TC

In general there are four ways how the assignments of two generics into the two sets of operators $\mathbb{A}$ (of Alice) and $\mathbb{B}$ (of Bob) can be done. On $\mathcal{H}^\infty$ this results in the following possibilities

$$\text{Given } \mathcal{A}^{M[r]} \text{ then } \mathcal{B}^{M[c]} \text{ is excluded;}$$
$$\text{Given } \mathcal{A}^{M[r]} \text{ then } \mathcal{B}^{M[r]} \text{ is not excluded;}$$
$$\text{Given } \mathcal{A}^{M[c]} \text{ then } \mathcal{B}^{M[r]} \text{ is excluded;}$$
$$\text{Given } \mathcal{A}^{M[c]} \text{ then } \mathcal{B}^{M[c]} \text{ is not excluded.}$$

While on the product space $\mathcal{H}_a \otimes \mathcal{H}_b$ the corresponding possibilities read

$$\text{Given } \mathcal{A}^{M[r]} \text{ then } \mathcal{B}^{M[c]} \text{ is not excluded;}$$
$$\text{Given } \mathcal{A}^{M[r]} \text{ then } \mathcal{B}^{M[r]} \text{ is not excluded;}$$
$$\text{Given } \mathcal{A}^{M[c]} \text{ then } \mathcal{B}^{M[r]} \text{ is not excluded;}$$
$$\text{Given } \mathcal{A}^{M[c]} \text{ then } \mathcal{B}^{M[c]} \text{ is not excluded.}$$

These cases repeat the part of the argument in the proof of Theorem 4 in the broadest context.

4.4 ITPs and sequences of QM measurements

Here we follow the original proof in ref. Landsman 2020 of 1-randomness and the construction of the Born's probability on infinite sequences of QM outcomes. This analysis contains the ITPs of Von Neumann and the explicit use of quantum states which were missing in the main body of the paper. This construction deals with the measurement procedure on ITP and the infinite sequences of measurements.

Let H be a finite dimensional complex Hilbert space and $B(H)$ the algebra of all bounded operators on $\mathcal{H}$. Then $a = a^*, a \in B((H)$ means a is self-adjoint. First it is considered the following set of commuting operators $a_1, \cdots, a_N$ on the finite tensor product Hilbert space $\mathcal{H} = \bigotimes_{i=1}^N H_i$ where all $H_i = H, i = 1, \cdots, N$ (we can take $H_i = \mathbb{C}^2, i = 1, 2, \cdots, N$)

$$[a_1 = a \otimes 1_H \otimes \cdots \otimes 1_H]; \cdots; [a_N = 1_H \otimes \cdots 1_H \otimes a]. \tag{9}$$

Having fixed such elementary setup we can extend this up to infinite products of Hilbert spaces and consider Born's probability on the resulting space. Still we follow ref. Landsman 2020. The operator $a \in B(H)$ determines the maximal commutative C^*-algebra, $C^*(a)$, which is isomorphic to the algebra of continuous functions on the spectrum $s(a)$, $C^*(a) \simeq C(s(a))$. In the finite dimensional case $s(a)$ contains the eigenvalues of a, i.e. $\lambda_a \in s(a), a\psi = \lambda_a\psi,\ \psi \in H$. For the family (9) of commuting operators, $\mathbf{a}$, there is the joint spectrum $S(\mathbf{a})$ and the C^*-algebra generated by the operators, $C^*(\mathbf{a})$ which still has its isomorphic representation, $C^*(\mathbf{a}) \simeq C(S(\mathbf{a}))$ where now $S(\mathbf{a})$ contains all nonzero eigenvalues $\lambda(\mathbf{a}) = (\lambda_1, \cdots, \lambda_N)$ for the same joint eigenvector $\psi \in \mathcal{H}$, i.e. $a_1\psi = \lambda_1\psi, \cdots, a_N\psi = \lambda_N\psi$. Let $e_{\lambda_i}, i = 1, \cdots, N$ be the projections of a_i on the one-dimensional subspaces corresponding to λ_i in H and $\omega_{\mathbf{a}}$ a state on $B(\mathcal{H})$ determining the Born's join probability of finding $\lambda(\mathbf{a})$ in a measurement, which reads (provided $e_{\lambda(\mathbf{a})}$ is the projection of $\mathbf{a}$ on $e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_N} \neq 0$)

$$p(\lambda(\mathbf{a})) = \omega(e_{\lambda(\mathbf{a})}).$$

For the finite dimensional case this reduces to the probability determined by a density operator ρ, i.e. $p(\lambda(\mathbf{a})) = \mathrm{Tr}(\rho(e_{\lambda(\mathbf{a})}))$. Representing the density operator corresponding to a pure state as $\rho = |\psi\rangle\langle\psi|$

for a unit vector $\psi \in \mathcal{H}$, we obtain the Born's formula for the joint probability

$$p(\lambda(\mathbf{a})) = \langle \psi | (e_{\lambda(\mathbf{a})}) \psi \rangle.$$

Since ω is a state on $B(\mathcal{H})$ and the measurement of $\mathbf{a}$ is performed on the (system with) N-tensor product space in the state ω such that this is generated by the N-fold measurements of $a \in B(H)$ on H, as in (9), in the state $\omega_1 \in B(H)^*$. It should hold

$$\omega = \omega_1^N \text{ which acts on the } N\text{-fold tensor product of vectors by}$$

$$\omega_1^N (e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_N}) = \omega_1(e_{\lambda_1}) \cdots \omega_1(e_{\lambda_N}) = p(\lambda_1) \cdots p(\lambda_N).$$

This is the joint probability in the measurement for the entire run of the finite sequence of N measurements. Next let us extend the procedure over the infinite sequences. The strict relation between the Born's probability of a single measurement of a and the Born's probability of the whole-run of the infinitely many measurements as in $\mathbf{a}$, holds true, provided the von Neumann complete ITPs are applied. Following ref. Landsman 2020 there are crucial regularity properties of the Nth fold tensor product of the spectral C^*-algebras

$$1)\ B(H)^{\otimes N} = B(H \otimes \cdots \otimes H); \quad 2)\ C^*(a)^{\otimes N} = C^*(a_1, \cdots, a_N), \text{ for } a \in B(H), a_1, \cdots, a_N \in B(\mathcal{H} = H^{\otimes N}).$$

One way to make sure that the limiting cases: $\lim_{N \to \infty} H^{\otimes N}$ and $\lim_{N \to \infty} B(H)^{\otimes N}$ are the complete ITP H^∞ of von Neumann and $B(H^\infty)$, correspondingly, is by taking the class of infinite sequences $\mathbf{a} = (a_1, a_2, \cdots)$ such that for any such sequence there exists $M \in \mathbb{N}$ and $a_M \in C^*(a)^{\otimes M}$ such that for any $N > M$, $a_N = a_M \otimes 1_{C^*(a)} \otimes \cdots \otimes 1_{C^*(a)}$. Based on this condition one shows that

$$\lim_{N \to \infty} H^{\otimes N} = H^\infty \text{ the complete ITP, and}$$

$$\lim_{N \to \infty} B(H)^{\otimes N} = B(H^\infty), \text{ the bounded operators on the complete ITP } H^\infty,$$

$$\lim_{N \to \infty} C^*(a)^N = C^*(a)^\infty = C(s(a))^{\mathbb{N}} = C^*(a_1, a_2, \cdots).$$

This last case corresponds to the infinite product of compact measure spaces (note that here we have $\mathbb{N}$ - the set of all natural numbers). Also in this infinite case given the state ω_1 on $B(H)$ one uniquely determines the state ω_1^∞ on $B(H)^{\otimes \infty} = B(H^\infty)$. Finally, the measure $\mu_{\mathbf{a}}$ derived from the infinitely long sequences of operators, $\mathbf{a}$, is determined by the one element sequence, a (see (9)), i.e.

$$\mu_{\mathbf{a}} = \mu_a^\infty.$$

There are other possible constructions of ITPs, like grounded tensor product of countably many grounded Hilbert spaces Baez, Segal, and Zhou 1992 or inductive limits of the inductive family of the finite subsets of indices set Guichardet 1992. These both examples are more physics directed since the resulting ITPs are already separable Hilbert spaces. However, they can be also realized as closely related to the truncated von Neumann ITP by taking suitable set of infinite sequences $\mathbf{a}$.

4.5 ZFC twist of QM

This supplementary section explains the relation of the standard QM with QM where infinite sequences of outcomes are explicitly Solovay random. The ZFC randomness of infinite sequences is, in fact, formally present already in the Born's probability and this presence becomes transparent under additional specification of the set theory side of QM. This is rather a change of the point of view to the complementary to quantum logic one, than the change of the standard QM formalism.

Below we will demonstrate briefly how the sets perspective emerges from logical one in QM on Hilbert spaces. Taking the set theory completion of QM directly refers to ZFC randomness. In theorem 4 in the main body of the paper we have assumed the ZFC randomness of QM and then found the way how to negate TC. This section shows that ZFC randomness is already a part of QM formalism when performing the set theory completion.

Solovay randomness requires infinite dimension of Hilbert space. Let $\dim \mathcal{H} = +\infty$ and $\mathbb{L}(\wedge, \vee, 0, 1)$ the lattice of projections on $\mathcal{H}$ then the maximal complete Boolean algebras chosen from it all have the isomorphic atomless part which is the measure algebra B. This was explained already at the beginning of the Subsection 4.1. Thus QM on infinite dimensional Hilbert space is always paired with a quantum logic carried by the lattice structure with the B as above. Our concern here is quantum set theory which is also determined by $\mathcal{H}^{(\infty)}, \mathbb{L}(\wedge, \vee, 0, 1)$ and B but this requires some additional clarification. First of all, is there any well-defined meaning assigned to something like *quantum set theory*. The answer can encompass the variety of proposals, though one is especially natural from the formal point of view. This follows the logic of QM on $\mathbb{L}$ and the way how Boolean-valued models of ZFC are construed. Let us start with 2-valued Boolean models of ZFC (where $2 = \{0, 1\}$ is 2-valued Boolean algebra). The connection of this 2-valued logic with models of ZFC is given trivially as $V \simeq V^2$ or $M \simeq M^2$, where V and M are class model and set model of ZFC, correspondingly. Thus 2-valued models of ZFC are (isomorphic images of) the models. The next step would be V^B (M^B) for arbitrary complete Boolean algebra B in V (M) replacing the algebra 2 above. Such Boolean-valued models of ZFC lie in the heart of Boolean-valued analysis which has been intensively developed since its invention by Gaisi Takeuti, Dana Scott and Robert Solovay in 1970ties. The models V^B have been also exploited in the main body of this paper where B was the complete atomless measure algebra $Bor(\mathbb{R})/\mathcal{N}$ with the relations between the models $V^B \hookrightarrow V \hookrightarrow V^B$. The next step is the full quantum set theory. Takeuti Takeuti 1978 proposed it to be the q-universe of sets, i.e. the lattice-valued model $V^{\mathbb{L}}$ in analogy with the Boolean-valued models V^B. However, $V^{\mathbb{L}}$ is very complicated and seemed to Takeuti untractable as a resonable universe of sets (though see the recent development by Ozawa M. Ozawa 2021). However, the family of Boolean-valued models $\{V^B\}$ is a good approximation of $V^{\mathbb{L}}$ as each V^B corresponds to the local Boolean contexts of quantum theory. On the other side the contexts are given by maximal families of commuting observables. The entire lattice model $V^{\mathbb{L}}$ can be engulfed, or covered, by Boolean models V^B but the family of such V^Bs has to be augmented by some class of relations between them. The precise structure of the relations is not relevant here and we can express the following correspondences on $\mathcal{H}^{(\infty)}$ between logic, sets and operators

$$\{\text{logical local contexts of QM}\} \longleftrightarrow \{\text{measure Boolean algebras, } Bs\}$$

$$\{\text{maximal algebras of commuting observables in QM}\} \longleftrightarrow \{\text{measure Boolean algebras, } Bs\}$$

$$\{\text{set theoretic local contexts of QM}\} \longleftrightarrow \{\text{Boolean-valued models of ZFC, } V^B s\}$$

$$\text{global set theory of QM} \longleftrightarrow \text{non-Boolean, non-Heyting universe of sets } V^{\mathbb{L}}.$$

This last follows from the observation that each maximal Boolean algebra $B \subset \mathbb{L}$ leads to $V^B \subset V$ which is a submodel of the quantum set theory universe $V^{\mathbb{L}}$. The second correspondence expresses the fact that for any family of commuting self-adjoint observables there always exists a maximal Boolean algebra B of projections determining all the operators (the spectral families of these self-adjoint operators have values in projections from B, e.g. ref. Takeuti 1978).

So far we have the universe $V^{\mathbb{L}}$ approximated by the family $\{V^B\}$ of Boolean-valued ZFC universes as local contexts. It remains to describe the step from Boolean many-valued to 2-valued contexts. This is also where Solovay randomness emerges. In general one refers to ZFC models M with the internal measure algebra B_M and builds in M the Boolean-valued model $M^{B_M} \subset M$ of ZFC. Let us use M^B for this M^{B_M} as was the case in the main body of the paper. This M^B is the carrier of the Solovay randomness via the forcing extension $M[r] \simeq M^B/Ult_r$ whre r is a Solovay random real. General results in set theory indicate that one has nontrivial forcing extension of the model M whenever the algebra B is complete and atomless. This is precisely the case for the measure algebra $Bor(\mathbb{R})/\mathcal{N}$ determined by the QM lattice $\mathbb{L}$. Thus the way from the non-distributive logic of QM and Boolean contexts V^Bs to the 2-valued world with 2-valued logic and set theory goes in infinite dimension, via random forcing extensions $V[r_\alpha]$, $\alpha \in I$, since these last models are already 2-valued and standard. This formal reasoning is a part of mathematics of QM and even though it bears the attribute of formal necessity, one can still ask whether QM as physical theory realizes such scenario. We already know that a single run of quantum measurement performed on a system with $\mathcal{H}_i$ has the property that its Born's measure determines the unique probability measure on the space of infinite repetitions on $\bigotimes_{i\in I} \mathcal{H}_i$, such that the 1-randomness of the sequences is maintained by the single run Born's rule. Does this single run on $\mathcal{H}_i$ determine the Solovay genericity of the outcomes?

In general one can construct the formalism of QM in models of ZFC, or in the universe V, this is also true for majority of mathematical constructions. QM is not any exception in this regard (though see ref. Benioff 1976a, 1976b where it has been shown that not all models are equally good). Moreover, any ZFC provable property is valid in every model of ZFC. Thus if the observer would use ZFC-provable techniques they remain the same in every universe of set and their richness does not matter since the models are indistinguishable (or simply there are no models). Set theory of QM indicates rather different behavior regarding the states: they might belong to different models though this is not observable, so the Born's probability is not affected. The variety of Boolean-valued universes is present in the QM formalism which follows the set theory perspective.

The following statement is the condition for the outcomes of QM measurements be ZFC random (Solovay generic). This is a reformulation of the generic measurement property discussed in Subection 4.2 and was already referred to, in a bit different context, in ref. Król, Bielas, and Asselmeyer-Maluga 2023

> *The Measurement Postulate of q set theory.* The Born measure of a single quantum run determines the Solovay genericity of infinite sequences of QM runs iff there exists a model M of ZFC such that QM measuring process of the quantum system admits description in M, i.e. the quantum state of the system before measurement is in M along with the preparing procedure for the mesurement, while after the measurement the quantum state of the system is in $M[r]$, where $r \in M[r]$ is a generic real and has the binary development representing the infinite sequence of the QM outcomes of measurements.

In the standard QM the set theory perspective is usually missing. The set theory perspective is rather the change of the point of view (the twist) in QM than the true change of the standard QM. Quantum set theory is complementary to quantum logic in the standard QM formalism.

Still remains unanswered whether the Solovay randomness (ZFC randomness in the terminology of ref. Król, Bielas, and Asselmeyer-Maluga 2023) can really be seen in experiments directly. This is quite analogous to the common use of infinite dimensional Hilbert spaces in QM while any discrimination of finite and infinite dimensional cases experimentally is not possible at present Coladangelo and Stark 2020.

Notes

Competing Interests None.

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