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Accelerated Expansion of the Universe in Terms of Differential Spaces

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Abstract

The concept of closures of manifolds in the category of Sikorski's differential spaces is applied to a description of the flat FRW models. The smoothness condition coming from this approach constitutes a strong restriction on the time dependence of the scale factor and on the energy density of the matter content of the resulting model. We demonstrate that our model agrees with the $H(z)$ dependence obtained with the help observational data concerning the type Ia supernovae, BAO and the CMB peaks tests. The model contains a string gas, two types of domain walls, four types of cosmological vacuums and a cosmological constant whose value — determined by the model — agrees with the results of the Planck Mission.

Keywords: accelerated cosmological models, differential spaces

1. Introduction

Almost at the beginning of the general relativity theory researchers noticed problems connected with the initial singularity in cosmology. There have been several attempts to overcome these problems. Below we list a few of them.

The first approach is based on the assumption that there is a possibility to construct a classical gravity theory that is free of singularities. In this strategy the potential alternative theory of gravity must contain GR as the weak field limit Einstein 1945, 1948, 1955. One of the proposals of such a theory is based on non-Riemannian geometry Cornish and Moffat 1994,Damour, Deser, and McCarthy 1993,Dobrowolski and Koc 2015 in which the metric tensor can be split into the symmetric part and the skew-symmetric part. From the mathematical point of view non-Riemannian geometry enables to circumvent the assumptions of the Hawking – Penrose singularity theorems Hawking and Penrose 1970.

The second approach involves noncommutative geometry Connes 1994; Madore 1999; Gracia-Bondía, Várilly, and Figueroa 2001; Heller, Sasin, and Lambert 1997; Heller, Pysiak, and Sasin 2005; Heller et al. 2015. This vast branch of modern mathematical physics aims at reconstructing the differential-geometrical notions *more algebraico* (most notably, in the language of Connes' spectral triples), and then, by abandoning the requirement that the algebras involved be commutative, it is believed to provide a unified mathematical framework for the study of both relativity and quanta. In doing so, the very notion of a space-time point is replaced with a more structuralized, global object, and the troublesome singularities no longer appear.

In the third approach it is presumed that all singularities disappear at the more fundamental, quantum level of the gravity theory Rosenfeld 1930b, 1930a; Rovelli and Smolin 1995a, 1995b; Markopoulou and Smolin 1998; Barrett and Crane 1998; Baez and Crane 1998, 1999. It is believed that the relation between quantum gravity and general relativity is similar to the relation between classical hydrodynamics (i.e. a theory that admits singularities) and the microscopic description of a fluid which, due to the finite size of the molecules, is free of singularities.

However, one might perceive the initial singularity not as a theoretical obstacle to be overcome, but rather as a real feature of the Universe. The main stream of research based upon this view comprises the theories of singular boundaries: g-boundaries, b-boundaries, c-boundaries, a-boundaries R. Geroch 1971, R. P. Geroch 1968, Schmidt 1971, Geroch, Kronheimer, and Penrose 1972, Geroch and Horowitz 1979, Scott and Szekeres 1994 and others.

The method approach we adopt in the present paper in the same vein as the last of the above-mentioned ones. Since the category of manifolds is a subcategory of the category Sikorski's differential spaces (called d-spaces for short) , therefore one can define closures of the flat Friedmanian model manifolds which are d-spaces Sikorski 1967, Sikorski 1971, Sikorski 1972, Waliszewski 1972, Sasin, Heller, and Multarzyński 1989, Gruszczak, Heller, and Multarzynski 1988, Gruszczak and Heller 1993. As a final result we obtain the so-called smoothness equation which we discuss in Section 2. However, this discussion will be descriptive. We do not want to burden this paper with an excessive mathematical abstraction, but rather we want to concentrate on results that may have meaning in cosmology.

A method for solving the smoothness equation is presented in Section 3. The explicit form of the solutions of the smoothness equation is shown in Section 4 (see also Gruszczak 2014). Discussion on the matter content of our model is carried out in Section 5. In Section 6 we compare our model with the observational data. The final discussion and summary are provided in Section 7.

2. On the smoothness equation

We restrict our considerations to the homogeneous, isotropic and flat cosmological models described by the FRW metric

$$g = c^2 dt^2 - a^2(t) \left(dx^2 + dy^2 + dz^2 \right), \tag{1}$$

where $a(t)$ is the scale factor, t the cosmological time. In addition, we assume that the model has an initial singularity at $t = 0$, $a(t = 0) = 0$, and starts its evolution with the velocity $\dot{a}(t = 0) > 0$.

The serious problem of the FRW cosmological models is that their manifold structures break down at $t = 0$ and therefore this moment cannot be included in the description of their time evolution. The solution we propose below is to perform the closure of the flat cosmological models in the class of differential spaces, objects more general than manifolds. This enables one to prolong the time orientability notion to the edges of the closures –called the *differential closures* (*d-closures*)– and thus to include the moment $t = 0$ into our investigations. Let us emphasize that this prolongation cannot be, in general, realized by means of the singular boundaries from the classical theory of singularities (see Heller et al. 1992). Our method was discussed in detail in Gruszczak 2014.

Including the beginning of time t in the above-described way imposes the following restrictive condition, called the *smoothness equation*

$$\dot{a}(t) = f\left(a(t), a(t) \int_0^t \frac{d\tau}{a'(\tau)} \right), \quad a(t = 0) = 0, \quad f(0, 0) > 0, \tag{2}$$

where $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is a function such that the physical dimensions of the left and right sides of the smoothness equation are the same. The condition $f(0, 0) > 0$ comes from the physical assumption that our Universe starts its expansion with positive "velocity".

For example, if we assume that f is an affine function of variables $f(x, y) = \beta + \gamma_1 x + \gamma_2 y$, where $(x, y) \in \mathbb{R}^2$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $\beta > 0$ then the smoothness equation (2), after rescaling, takes the form

$$\dot{\bar{a}}(t) = 1 + \gamma_1 \bar{a}(t) + \bar{\gamma}_2 \bar{a}(t) \int_0^t \frac{d\tau}{\bar{a}'(\tau)}, \quad \bar{a}(t = 0) = 0, \tag{3}$$

where $\bar{a}(t) := a(t)/\beta$ and $\bar{\gamma}_2 := \gamma_2/\beta$. Equation (3) will be called *the simplest smoothness equation*. Cosmological models (1) with scale factors satisfying the smoothness equation we shall call *the models evolving smoothly from the very beginning* or *the smoothly evolving models* or *the SE-models* for brevity.

The smoothness equation was introduced in Gruszczak 2014. On the physical side, it guarantees that for models (1) satisfying the equation the time orientability given by the vector field representing the cosmological time t can be smoothly prolonged to the moment $t = 0$. It is worth adding that in the theory of d-spaces the smoothness notion is more general than in the theory of manifolds.

3. The method of solving the simplest smoothness equation

In order to solve the smoothness equation (3), let us introduce the following auxiliary function

$$\bar{v}(t) = \int_0^t \frac{d\tau}{\bar{a}'(\tau)}. \tag{4}$$

Since in model (1) one has $\dot{\bar{a}}(t=0) > 0$, the domain $[0, t_f]$ of $\bar{v}(t)$ is defined by the condition $\dot{\bar{a}}(t) \geqslant 0$. Depending on the values of γ_1 and $\bar{\gamma}_2$, the value of t_f is either finite or infinite. The $\bar{v}(t)$ is an increasing function in its domain and therefore it has the inverse function $t(\bar{v})$. It is worth noticing that, in a similar vein, the assumption $\dot{\bar{a}}(t) > 0$ ensures the existence of the function inverse to $\bar{a}(t)$ denoted as $t(\bar{a})$.

Now, we can rewrite the smoothness equation in the form featuring $\bar{v}(t)$

$$\dot{\bar{a}}(t) = 1 + (\gamma_1 + \bar{\gamma}_2 \bar{v}(t))\bar{a}(t). \tag{5}$$

After the change of variables $\dot{\bar{a}}(t) = d\bar{a}(\bar{v})/d\bar{v}|_{\bar{v}=\bar{v}(t)} \cdot d\bar{v}(t)/dt = d\bar{a}(\bar{v})/d\bar{v}|_{\bar{v}=\bar{v}(t)} \cdot \dot{\bar{a}}(t)^{-1}$ we obtain the formula

$$\dot{\bar{a}}(t)^2 = d\bar{a}(\bar{v})/d\bar{v}|_{\bar{v}=\bar{v}(t)}$$

which enables one to write the smoothness equation in the following useful form

$$d\bar{a}(\bar{v})/d\bar{v} = (1 + \bar{a}(\bar{v})(\gamma_1 + \bar{\gamma}_2\bar{v}))^2, \quad \bar{a}(\bar{v} = 0) = 0. \tag{6}$$

This equation is solvable by elementary methods. Its solution depends on the two external parameters γ_1 and $\bar{\gamma}_2$ coming from the smoothness equation (3).

This enables us also to set down the relation between the variables t and $\bar{v}$ or, in other words, to find the function $t(\bar{v})$ inverse to the function $\bar{v}(t)$. Indeed, since $d\bar{v}/dt = 1/\dot{\bar{a}}(t)$ therefore the inverse function satisfies

$$dt(\bar{v})/d\bar{v} = \dot{\bar{a}}(t)|_{t=t(\bar{v})} =: \dot{\bar{a}}(\bar{v}). \tag{7}$$

The function $\dot{\bar{a}}(t)$ is given by the smoothness equation (3). Its value at $t = t(\bar{v})$ is $\dot{\bar{a}}(\bar{v}) = 1 + \bar{a}(\bar{v})(\gamma_1 + \bar{\gamma}_2\bar{v})$. Thus, equation (7) takes the following new form

$$dt(\bar{v})/d\bar{v} = 1 + \bar{a}(\bar{v})(\gamma_1 + \bar{\gamma}_2\bar{v}). \tag{8}$$

The scale factor $\bar{a}(\bar{v})$ is a known function. It is the solution of equation (6). Therefore, equation (8) is integrable and its solution with the initial condition $t(\bar{v} = 0) = 0$ has the form

$$t(\bar{v}) = \int_0^{\bar{v}} (1 + \bar{a}(\bar{v}')(\gamma_1 + \bar{\gamma}_2\bar{v}'))d\bar{v}'. \tag{9}$$

It is worth to notice that the pair $(\bar{a}(\bar{v}), t(\bar{v}))$ is a parametric solution of the smoothness equation (3), where $\bar{a}(\bar{v})$ satisfies equation (6) and $t(\bar{v})$ is given by formula (9).

4. Solutions of the simplest smoothness equation

Flat cosmological models with scale factors satisfying the simplest smoothness equation were studied in Gruszczak 2014. In the case $\gamma_1 < 0$ and $\gamma_2 > 0$ the SE-model turned out to exhibit an interesting evolution which is qualitatively consistent with the results of observations of type Ia supernovae. Therefore, let us restrict further considerations to SE-models with the parameters from the range $\gamma_1 \leq 0$ and $\bar{\gamma}_2 \geq 0$.

The smoothness equation written in the form (6) is of the Riccati type. Its solution with the initial condition $\bar{a}(\bar{v} = 0) = 0$ reads

$$\bar{a}(\bar{v}) = \frac{1}{\sqrt{\bar{\gamma}_2}\coth(\sqrt{\bar{\gamma}_2}\,\bar{v}) - \gamma_1 - \bar{\gamma}_2\bar{v}}, \quad \bar{\gamma}_2 > 0 \tag{10}$$

and in the case $\bar{\gamma}_2 = 0$ it is given by the formula

$$\bar{a}(\bar{v}) = \frac{\bar{v}}{1 - \gamma_1\bar{v}}. \tag{11}$$

Let us notice that in the case $\bar{\gamma}_2 > 0$ the final moment of evolution corresponds to $\bar{a} \to \infty$. Therefore, the final value $\bar{v}_f$ of the parameter $\bar{v}$ is the solution of the following equation

$$\sqrt{\bar{\gamma}_2}\coth(\sqrt{\bar{\gamma}_2}\,\bar{v}_f) - \gamma_1 - \bar{\gamma}_2\bar{v}_f = 0. \tag{12}$$

Thus, in this case $\bar{v} \in [0, \bar{v}_f)$.

When $\bar{\gamma}_2 = 0$ the domain of $\bar{a}(\bar{v})$ is the set $[0, \infty)$.

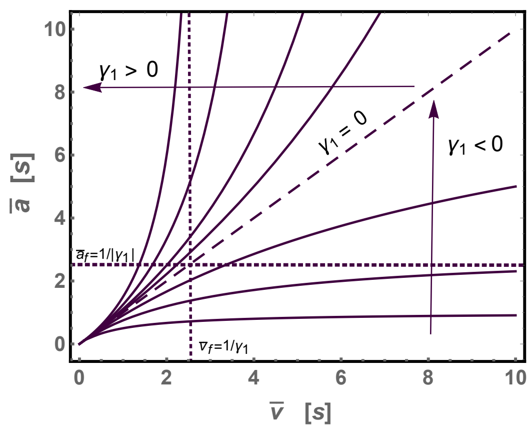

Figure 1. Scale factor $\bar{a}(\bar{v})$ for the SE-model with $\gamma_2 = 0$ and $\gamma_1 \in \mathbb{R}$. When $\gamma_1 = 0$ the model expands with a constant velocity. For $\gamma_1 > 0$ the SE-model accelerates from the very beginning while for $\gamma_1 < 0$ it decelerates and $\bar{a}(\bar{v}) \to \bar{a}_f = 1/|\gamma_1|$ when $\bar{v} \to \infty$. In the last case the model becomes the Minkowski space-time in the final stage of its evolution .

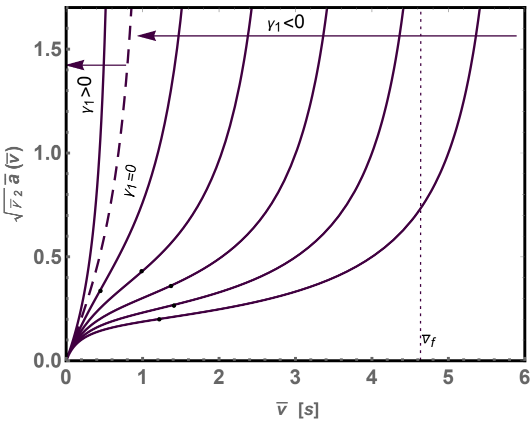

Figure 2. The re scaled scale factor $\sqrt{\gamma_2}\,\bar{a}(\bar{v})$ of the SE-model in the case $\gamma_2 > 0$ and $\gamma_1 \in \mathbb{R}$. Every curve on the plot has a vertical asymptote at $\bar{v} = \bar{v}_f$, where $\bar{v}_f$ satisfies equation (12). When $\gamma_1 \geqslant 0$ the model accelerates from the very beginning. For $\gamma_1 < 0$ the model initially decelerates but at moments indicated by small black points on the graph an accelerated evolution commences.

5. On the matter content of the SE-model

In the standard cosmology one employs the following methodological scenario. One assumes what kind(s) of fluid(s) permeate(s) the cosmological model and then, on this basis, deduces the evolutionary properties of the cosmological model solving Friedman's equations. In the case of the flat models, which we shall discuss in the current paper, the equations read

$$\tilde{\varepsilon}(t) := \frac{3}{\kappa c^2}\frac{\dot{a}(t)^2}{a(t)^2}, \qquad \tilde{p}(t) := -\frac{1}{\kappa c^2}\left(2\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}(t)^2}{a(t)^2}\right), \tag{13}$$

$$\tilde{\varepsilon} = \varepsilon + \Lambda/\kappa, \qquad \tilde{p} = p - \Lambda/\kappa, \tag{14}$$

where $\kappa = 8\pi G/c^4$, Λ is the cosmological constant and ε and p denote the energy density and pressure of all assumed kinds of fluids filling up the model investigated. The form of $\tilde{\varepsilon}$ and $\tilde{p}$ in formula (14) depends on our choice. We can work with or without the cosmological constant Λ. If we decide to work with a nonzero cosmological constant then the Λ appears in the solutions of equations (13) as an additional parameter.

In the present paper we reverse the standard methodological scenario outlined above. We assume that our model evolves from the initial singularity according to solutions (10) or (11). On this basis we try to reconstruct the matter content of the SE-model with the help of equations (13). Now $\tilde{\varepsilon}$ and $\tilde{p}$ are known functions defined by the right-hand sides of equations (13). They do not depend on any additional parameters, in particular on Λ. Therefore, in the present context, we cannot simply choose Λ to be zero or non-zero. However, we can still employ physical argumentation. Concretely, it is reasonable to assume that all forms of the usual matter should 'disperse' as the Universe expands to infinite size. In that case, the energy density ε and pressure p of every kind of fluid should vanish. This requirement can be expressed as follows: $\lim_{a\to\infty}\varepsilon(a) = 0$, $\lim_{a\to\infty} p(a) = 0$. Therefore, if only limits of our $\tilde{\varepsilon}(a)$ and $\tilde{p}(a)$ satisfy the following condition

$$\lim_{a\to\infty}\tilde{\varepsilon}(a) = -\lim_{a\to\infty}\tilde{p}(a) \neq 0 \tag{15}$$

we can define, with the help of formulas (14), the cosmological constant Λ_{th} in our model

$$\Lambda_{\text{th}} := \kappa \lim_{a\to\infty}\tilde{\varepsilon}(a). \tag{16}$$

Let us return to the discussion of the SE-model. First we notice that the Hubble function can be obtained with the help of the smoothness equation (3)

$$H(t) := \dot{\bar{a}}(t)/\bar{a}(t) = 1/\bar{a}(t) + \gamma_1 + \bar{\gamma}_2\bar{v}(t). \tag{17}$$

Then equations (13) for solutions of the smoothness equation can be rewritten in the form

$$\tilde{\varepsilon}(t) = 3H(t)^2/\kappa c^2, \tag{18}$$

$$\tilde{p}(t) = \frac{2}{\kappa c^2\bar{a}(t)}[H(t) - \bar{\gamma}_2/H(t)] - 3H(t)^2/\kappa c^2. \tag{19}$$

Since the solutions of the smoothness equation are known functions, the right-hand sides of formulas (18) and (19) are treated now as the definitions of $\tilde{\varepsilon}(t)$ and $\tilde{p}(t)$ respectively.

Now, $H(t)$ can be expressed in the form dependent on $\bar{a}$

$$H(\bar{a}) := H(t)|_{t=t(\bar{a})} = 1/\bar{a} + \gamma_1 + \bar{\gamma}_2\bar{v}(\bar{a}), \tag{20}$$

where $\bar{v}(\bar{a}) := \bar{v}(t)|_{t=t(\bar{a})}$ is the inverse function of the known scale factor $\bar{a}(\bar{v})$ (see formulas (10) and (11)). This form of H is suitable for the following discussion.

Thus, also the $\tilde{\varepsilon}$ in our model can be presented in the form depended on the variable $\bar{a}$

$$\tilde{\varepsilon}(\bar{a}) := 3H(\bar{a})^2/\kappa c^2 = \left[\frac{3}{\bar{a}^2} + \frac{6\gamma_1}{\bar{a}} + 3\gamma_1^2 + 3\bar{\gamma}_2^2\bar{v}(\bar{a})^2 + \frac{6\bar{\gamma}_2\bar{v}(\bar{a})}{\bar{a}} + 6\gamma_1\bar{\gamma}_2\bar{v}(\bar{a})\right]/\kappa c^2. \tag{21}$$

This expression, however, not yet ready redy for interpretation because the function $\bar{v}(\bar{a})$ is not given in an explicit form. Nevertheless, one can extract certain properties of $\bar{v}(\bar{a})$ in the neighbourhood of $\bar{a} = 0$ by means of the series expansion of $\bar{a}(\bar{v})$ at $\bar{v} = 0$. The first terms of the inverse series read

$$\bar{v}(\bar{a}) = \bar{a} - \gamma_1\bar{a}^2 + \frac{1}{3}(3\gamma_1^2 - 2\bar{\gamma}_2)\bar{a}^3 + \dots \quad . \tag{22}$$

This means that $\bar{v}(\bar{a})$ can be written as

$$\bar{v}(\bar{a}) = \bar{a}(1 + \psi(\bar{a})),$$

where the map ψ satisfies the condition $\psi(\bar{a} = 0) = 0$. Thanks to this observation we can set down the correct form of the vacuum term in formula (21)

$$\tilde{\varepsilon}(\bar{a}) = \left[\frac{3}{\bar{a}^2} + \frac{6\gamma_1}{\bar{a}} + 3(\gamma_1^2 + 2\bar{\gamma}_2) + 3\bar{\gamma}_2^2\bar{v}(\bar{a})^2 + 6\bar{\gamma}_2\psi(\bar{a}) + 6\gamma_1\bar{\gamma}_2\bar{v}(\bar{a})\right]/\kappa c^2. \tag{23}$$

To recognize fluids that fill up the SE-model we apply the traditional interpretation method which depends on the form of $\tilde{\varepsilon}(\bar{a})$ and the barotropic index w usually used in cosmology. Necessary formulas for $\tilde{p}(\bar{a})$ can be obtained with the help of equality (19) and the composition $\tilde{p}(\bar{a}) = \tilde{p}(t)|_{t=t(\bar{a})}$.

In order to guess the matter content of the SE-model with the late acceleration, $(\gamma_1 < 0, \bar{\gamma}_2 > 0)$, let us first discuss its behavior for γ-parameters satisfying: $(\gamma_1 = 0, \bar{\gamma}_2 = 0)$, $(\gamma_1 < 0, \bar{\gamma}_2 = 0)$ and $(\gamma_1 = 0, \bar{\gamma}_2 > 0)$.

Example 1 *The SE-model with* $\gamma_1 = 0$ *and* $\bar{\gamma}_2 = 0$.

In this case formulas (19–20) yield

$$\tilde{\varepsilon}(\bar{a}) = 3/\kappa c^2\bar{a}^2, \quad \tilde{p}(\bar{a}) = -1/\kappa c^2\bar{a}^2. \tag{24}$$

In this model there are no reasons to introduce the cosmological constant and therefore $\tilde{\varepsilon} = \varepsilon$ and $\tilde{p} = p$. Thus, the model is filled with a string gas with the equation of state $p/\varepsilon = -1/3$. The string gas causes an unlimited expansion of such a universe with the constant velocity $\dot{\bar{a}} = 1$ (see Figure (1)). It is worth to notice that cosmological models with a nonzero curvature exhibit a similar dependence ε of a ($\varepsilon \propto 3/a^2$). One can say that the string gas substitutes the curvature in flat cosmological models Dąbrowski and J. 1989; Dąbrowski 1996; Kamenshchik and Khalatnikov.

Example 2 *The SE-model with* $\gamma_1 < 0$ *and* $\bar{\gamma}_2 = 0$.

In this case formulas on effective energy density and pressure have the form

$$\tilde{\varepsilon}(\bar{a}) = \left(\frac{3}{\bar{a}^2} + \frac{6\gamma_1}{\bar{a}} + 3\gamma_1^2\right)/\kappa c^2, \tag{25}$$

$$\tilde{p}(\bar{a}) = \left(-\frac{1}{\bar{a}^2} - \frac{4\gamma_1}{\bar{a}} - 3\gamma_1^2\right)/\kappa c^2. \tag{26}$$

From the viewpoint of the traditional interpretation our model is filled with three types of fluids: a string gas, domain walls and a cosmological vacuum. These class of models were considered in papers Dąbrowski 1996; Dąbrowski and Larsen 1995. In what follows we shall call the domain walls and the vacuum the γ_1-*domain walls* and the γ_1-*vacuum* respectively. Let us notice that one can interpret the γ_1-domain walls term as a potential term since $\gamma_1 < 0$.

Let us look once more on the evolutionary properties of the SE-model shown in Figure 1. The scale factor $\bar{a}$ is an increasing function of the variable $\bar{v}$ and $\lim_{\bar{v}\to\infty} \bar{a}(\bar{v}) = 1/|\gamma_1| =: \bar{a}_f$. It means geometrically that such a universe asymptotically becomes the Minkowski space-time for every $\gamma_1 < 0$. This fact is mirrored in the behaviour of $\tilde{\varepsilon}(\bar{a})$ at $\bar{a}_f$

$$\lim_{\bar{a}\to\bar{a}_f} \tilde{\varepsilon}(\bar{a}) = \left(3\gamma_1^2 + 6\gamma_1|\gamma_1| + 3\gamma_1^2\right)/\kappa c^2 = 0^+. \tag{27}$$

It means that the γ_1-vacuum is not a passive vacuum but rather that it interacts with the string gas and the γ_1-domain walls causing that the final energy density and the final pressure to be zero. There are no reasons to introduce the cosmological constant in this model.

Example 3 *The SE-model with* $\gamma_1 = 0$ *and* $\bar{\gamma}_2 > 0$.

Now formulas on $\tilde{\varepsilon}(\bar{a})$ and $\tilde{p}(\bar{a})$ look as follows

$$\tilde{\varepsilon}(\bar{a}) = \left[\frac{3}{\bar{a}^2} + 6\bar{\gamma}_2 + 3\bar{\gamma}_2^2\bar{v}(\bar{a})^2 + 6\bar{\gamma}_2\psi(\bar{a})\right]/\kappa c^2, \tag{28}$$

$$\tilde{p}(\bar{a}) = \left[-1/\bar{a}^2 - 6\bar{\gamma}_2 - 3\bar{\gamma}_2^2\bar{v}(\bar{a})^2 - 4\bar{\gamma}_2\psi(\bar{a}) + 2\bar{\gamma}_2^2\bar{a}\bar{v}(\bar{a})/(1 + \bar{\gamma}_2\bar{a}\bar{v}(\bar{a}))\right]/\kappa c^2. \tag{29}$$

Analyzing subsequent terms in (28) one can see that the our model contains the string gas, a cosmological vacuum and an unknown fluid represented by the term $3\bar{\gamma}_2^2\bar{v}(\bar{a})^2/\kappa c^2$. The last term one can interpret as a potential energy density since $6\bar{\gamma}_2\psi(\bar{a})/\kappa c^2 \leq 0$.

An additional interpretation is provided by the comparison of $\tilde{\varepsilon}$ with $\tilde{p}$. The barotropic indexes w of the four terms in $\tilde{\varepsilon}$ and $\tilde{p}$ have the following values $-1/3$, -1, -1 and $-2/3$, respectively. It is a surprising fact that the unknown fluid represented by the term $3\bar{\gamma}_2^2\bar{v}(\bar{a})^2/\kappa c^2$ can be linked with a vacuum and that the potential energy density term $6\bar{\gamma}_2\psi(\bar{a})/\kappa c^2$ can be associated with domain walls. The last term in $\tilde{p}(\bar{a})$, which is a perturbation of Dalton's law, is a suggestion that there is an interaction between (some of) the fluids contained in the SE-model.

The fluids represented by the terms $3/\kappa c^2\bar{a}^2$, $6\bar{\gamma}_2/\kappa c^2$, $3\bar{\gamma}_2^2\bar{v}(\bar{a})^2/\kappa c^2$, $6\bar{\gamma}_2\psi(\bar{a})/\kappa c^2$ appearing in formula (29) will be called the *string gas*, the γ_2-*vacuum*, the γ_2^b-*vacuum* and the γ_2^b-*domain walls*, respectively. The superscript "b" refers to the fact that the fluid in question is defined by means of the barotropic index.

More information on our SE-model is provided by the asymptotic behaviour of the terms in the formulas for $\tilde{\varepsilon}(\bar{a})$ and $\tilde{p}(\bar{a})$

$$\tilde{\varepsilon}^f := \lim_{\bar{a}\to\infty} \tilde{\varepsilon}(\bar{a}) = \left(0 + \underline{6\bar{\gamma}_2} + 3\bar{\gamma}_2^2\bar{v}_f^2 - \underline{6\bar{\gamma}_2}\right)/\kappa c^2 = 3\bar{\gamma}_2^2\bar{v}_f^2/\kappa c^2, \tag{30}$$

$$\tilde{p}^f := \lim_{\bar{a}\to\infty} \tilde{p}(\bar{a}) = \left(0 - \underline{6\bar{\gamma}_2} - 3\bar{\gamma}_2^2\bar{v}_f^2 + \underline{4\bar{\gamma}_2} + \underline{2\bar{\gamma}_2}\right)/\kappa c^2 = -3\bar{\gamma}_2^2\bar{v}_f^2/\kappa c^2, \tag{31}$$

where $\tilde{\varepsilon}^f$ and $\tilde{p}^f$ denote the effective final energy density and the effective final pressure of matter. In formula (30) the γ_2-vacuum term is canceled by the γ_2^b-domain walls term while in formula (31) the γ_2-vacuum term is canceled by the γ_2^b-domain walls term and the term perturbing the Dalton's law. It indicates that between the γ_2-vacuum and the γ_2^b-domain walls there is an interaction. We have no suggestions that the string gas and the γ_2^b-vacuum are interacting fluids in the discussed mixture.

Let us notice that $\tilde{\varepsilon}^f$ and $\tilde{p}^f$ satisfy the following inequality $\tilde{\varepsilon}^f = -\tilde{p}^f = 3\bar{\gamma}_2^2\bar{v}_f^2/\kappa c^2 \neq 0$. Therefore we can introduce the cosmological constant $\Lambda_{\text{th}} = \kappa\tilde{\varepsilon}^f = 3\bar{\gamma}_2^2\bar{v}_f^2/c^2$ to our model. Plugging the above formulas for $\tilde{\varepsilon}$, $\tilde{p}$ and Λ_{th} into (14), we obtain

$$\varepsilon(\bar{a}) = \left[\frac{3}{\bar{a}^2} + 6\bar{\gamma}_2 + 3\bar{\gamma}_2^2(\bar{v}(\bar{a})^2 - \bar{v}_f^2) + 6\bar{\gamma}_2\psi(\bar{a})\right]/\kappa c^2, \tag{32}$$

$$p(\bar{a}) = \left[-1/\bar{a}^2 - 6\bar{\gamma}_2 - 3\bar{\gamma}_2^2(\bar{v}(\bar{a})^2 - \bar{v}_f^2) - 4\bar{\gamma}_2\psi(\bar{a}) + 2\bar{\gamma}_2^2\bar{a}\bar{v}(\bar{a})/(1 + \bar{\gamma}_2\bar{a}\bar{v}(\bar{a}))\right]/\kappa c^2. \tag{33}$$

Evidently, $\lim_{\bar{a}\to\infty}\varepsilon(\bar{a}) = \lim_{\bar{a}\to\infty}p(\bar{a}) = 0$. But the analogous limit for the barotropic index

$$\lim_{\bar{a}\to\infty} w(\bar{a}) = \lim_{\bar{a}\to\infty}\frac{p(\bar{a})}{\varepsilon(\bar{a})} = \lim_{\bar{v}\to\bar{v}_f}\frac{p(\bar{v})}{\varepsilon(\bar{v})} = -2/3$$

leads to an interesting conclusion that the final evolution stage of our model is dominated by the γ_2^b-domain walls.

The discussed SE-model filled with the string gas and the γ_2-fluids is subject to the accelerated expansion to infinity from the very beginning (see Figure 2).

Now, we are ready to discuss the matter content of the SE-model with the late acceleration (see Figure (2)).

Example 4 *The SE-model with* $\gamma_1 < 0$ *and* $\bar{\gamma}_2 > 0$.

Almost all properties of the previous examples are now cumulated. Formulas on $\tilde{\varepsilon}(\bar{a})$ and $\tilde{p}(\bar{a})$ have now the more complicated form

$$\tilde{\varepsilon}(\bar{a}) = \left(\frac{3}{\bar{a}^2} + \frac{6\gamma_1}{\bar{a}} + 3\gamma_1^2 + 6\gamma_1\bar{\gamma}_2\bar{v}(\bar{a}) + 3\bar{\gamma}_2^2\bar{v}(\bar{a})^2 + 6\bar{\gamma}_2 + 6\bar{\gamma}_2\psi(\bar{a})\right)/\kappa c^2, \tag{34}$$

$$\begin{aligned}\tilde{p}(\bar{a}) &= \left(-\frac{1}{\bar{a}^2} - \frac{4\gamma_1}{\bar{a}} - 3\gamma_1^2 - 6\gamma_1\bar{\gamma}_2\bar{v}(\bar{a}) - 3\bar{\gamma}_2^2\bar{v}(\bar{a})^2 - 6\bar{\gamma}_2 - 4\bar{\gamma}_2\psi(\bar{a})\right)/\kappa c^2 \\ &+ \frac{2}{\kappa c^2}\cdot\frac{\bar{\gamma}_2(\gamma_1 + \bar{\gamma}_2\bar{v}(\bar{a}))\bar{a}}{1 + (\gamma_1 + \bar{\gamma}_2\bar{v}(\bar{a}))\bar{a}}.\end{aligned} \tag{35}$$

Applying the traditional interpretation we can recognize four terms in formula (34), namely $3/\kappa c^2\bar{a}^2$, $6\gamma_1/\kappa c^2\bar{a}$, $3\gamma_1^2/\kappa c^2$ and $6\bar{\gamma}_2/\kappa c^2$. We can link these terms with the following fluids: the string gas, the γ_1-domain walls, the γ_1-vacuum and the γ_2-vacuum, respectively. The remaining terms can be interpreted with the help of the barotropic index w. The terms $-6\gamma_1\bar{\gamma}_2\bar{v}(\bar{a})/\kappa c^2$ and $3\bar{\gamma}_2^2\bar{v}(\bar{a})^2/\kappa c^2$ can be linked with cosmological vacuums which will be called the $\gamma_{1,2}^b$-*vacuum* and the γ_2^b-*vacuum*, respectively. The last undiscussed term we shall link with the γ_2^b-domain walls (see Example 3).

One can obtain some further properties of the fluids in question by considering the following limits

$$\begin{aligned} \tilde{\varepsilon}^f := \lim_{\bar{a}\to\infty} \tilde{\varepsilon}(\bar{a}) &= \left(0 - 0 + 3\gamma_1^2 + 6\gamma_1\bar{\gamma}_2\bar{v}_f + 3\bar{\gamma}_2^2\bar{v}_f^2 + \underline{6\bar{\gamma}_2} - \underline{6\bar{\gamma}_2}\right)/\kappa c^2 \\ &= 3(\gamma_1 + \bar{\gamma}_2\bar{v}_f)^2/\kappa c^2, \end{aligned} \tag{36}$$

$$\begin{aligned} \tilde{p}^f := \lim_{\bar{a}\to\infty} \tilde{p}(\bar{a}) &= \left(0 + 0 - 3\gamma_1^2 - 6\gamma_1\bar{\gamma}_2\bar{v}_f - 3\bar{\gamma}_2^2\bar{v}_f^2 - \underline{6\bar{\gamma}_2} + \underline{4\bar{\gamma}_2} + \underline{2\bar{\gamma}_2}\right)/\kappa c^2 \\ &= -3(\gamma_1 + \bar{\gamma}_2\bar{v}_f)^2/\kappa c^2. \end{aligned} \tag{37}$$

The underlined terms suggest that there is an interaction between the γ_2-vacuum and the γ_2^b-domain walls like in Example 3. The remaining terms in formulas (36) and (37) define nonzero $\tilde{\varepsilon}^f$ and $\tilde{p}^f$ such that $\tilde{p}^f = -\tilde{\varepsilon}^f$. If we assume that Λ (see formula (14)) is zero then the final energy density of all matter contained in the universe $\varepsilon^f \equiv \tilde{\varepsilon}^f > 0$. It means that in this case the universe explodes to infinity.

On the other hand, if we assume that $\Lambda \neq 0$ (see Example 3) then we can define the cosmological constant

$$\Lambda_{\text{th}} := \tilde{\varepsilon}^f = 3(\gamma_1 + \bar{\gamma}_2\bar{v}_f)^2/c^2. \tag{38}$$

The discussed SE-model with the cosmological constant $\Lambda = \Lambda_{\text{th}}$ will be called the *ΛSE-model*. For the ΛSE-model formulas on the energy density ε and pressure p have the form

$$\varepsilon(\bar{a}) = \left[\frac{3}{\bar{a}^2} + \frac{6\gamma_1}{\bar{a}} + V(\bar{a}) + 6\bar{\gamma}_2 + 6\bar{\gamma}_2\psi(\bar{a})\right]/\kappa c^2, \tag{39}$$

$$p(\bar{a}) = \left[-\frac{1}{\bar{a}^2} - \frac{4\gamma_1}{\bar{a}} - V(\bar{a}) - 6\bar{\gamma}_2 - 4\bar{\gamma}_2\psi(\bar{a}) + \frac{2\bar{\gamma}_2(\gamma_1 + \bar{\gamma}_2\bar{v}(\bar{a}))\bar{a}}{1 + (\gamma_1 + \bar{\gamma}_2\bar{v}(\bar{a}))\bar{a}}\right]/\kappa c^2, \tag{40}$$

where

$$V(\bar{a}) := 3(\gamma_1 + \bar{\gamma}_2\bar{v}(\bar{a}))^2 - 3(\gamma_1 + \bar{\gamma}_2\bar{v}_f)^2. \tag{41}$$

The term $V/\kappa c^2$ is a w-vacuum term. The fluid associated with this term shall be called the V^b-*vacuum*. It is a mixture of the γ_1-vacuum, the $\gamma_{1,2}^b$-vacuum and γ_2^b-vacuum. Since $V(\bar{a}) \leq 0$ for $\bar{a} \in [0, \infty)$, the $V(\bar{a})/\kappa c^2$ can be interpreted as a potential term in the energetic balance ε. Similarly to the result in Example 3

$$\lim_{\bar{a}\to\infty} w(\bar{a}) = \lim_{\bar{v}\to\bar{v}_f} \frac{p(\bar{v})}{\varepsilon(\bar{v})} = -2/3.$$

It means that the final evolution stage of the discussed model is dominated by the γ_2^b-domain walls.

6. Observational $H_{\text{obs}}(z)$ data and the ΛSE-model

In this section we present an observational motivation for the choice of the γ-parameters from the range $\gamma_1 < 0$ and $\bar{\gamma}_2 > 0$. First of all let us notice that in our ΛSE-model the dependence $H(z)$ can be expressed in a parametric form dependent on $\bar{v}$ because both the Hubble function $H(t)$ and the redshift $z(t) := \bar{a}(t_0)/\bar{a}(t) - 1$ can be written as functions of $\bar{v}$

$$H(\bar{v}) := H(t)|_{t=t(\bar{v})} = \sqrt{\bar{\gamma}_2}\coth(\sqrt{\bar{\gamma}_2}\bar{v}), \tag{42}$$

$$z(\bar{v}) := z(t)|_{t=t(\bar{v})} = \bar{a}(\bar{v}_0)/\bar{a}(\bar{v}) - 1, \tag{43}$$

where $H(t)$ is given by formula (17) and $\bar{a}(\bar{v})$ is a known function (10). The value of $\bar{v}$ at the present moment

$$\bar{v}_0 = \text{arccoth}(H_0/\sqrt{\bar{\gamma}_2})/\sqrt{\bar{\gamma}_2} \tag{44}$$

we can calculate with the help of the equality $H_0 = H(\bar{v}_0)$. The numerical value of $H_0 = 67.3\text{km}/(\text{s Mpc})$ is taken from the results of the Planck Mission *Planck Collaboration: Planck 2013 results. XVI. Cosmological parameters.*

Thus, the parametric equations for $H(z)$ can be written as

$$H(\bar{v}) = \sqrt{\bar{\gamma}_2}\coth(\sqrt{\bar{\gamma}_2}\bar{v}) \quad z(\bar{v}) = \bar{a}(\bar{v}_0)/\bar{a}(\bar{v}) - 1, \tag{45}$$

where $\bar{v} \in [0, \bar{v}_f)$ (see equation (12)).

Now, we can apply the χ^2 procedure in order to find the values of γ_1 and $\bar{\gamma}_2$ which best fit the SE-model to the recently updated observational data Zhang, Ma, and Lan 2010; Ma Cong 2011; Yu et al. 2011; Blake et al. 2012; Chuang and Wang 2013; Busca et al. 2013; Jimenez, Simon, and Verde 2005; Zhang et al. 2014; Moresco et al. 2012. As an outcome we obtain $\gamma_1 = -2.69921 \times 10^{-18}\,\text{s}^{-1}$ and $\bar{\gamma}_2 = 3.1211 \times 10^{-36}\,\text{s}^{-2}$ at the level of $\chi^2_{min} = 17.454$ (see Figure 3). For comparison, we used the result of the χ^2 procedure for the ΛCDM-model with the same data. The fit for the ΛCDM-model is represented by the dashed line while the prediction of the smoothly evolving model is represented by the solid line. The value of χ^2_{min} in this case is similar. Concretely, $\chi^2_{min} = 18.119$. The time

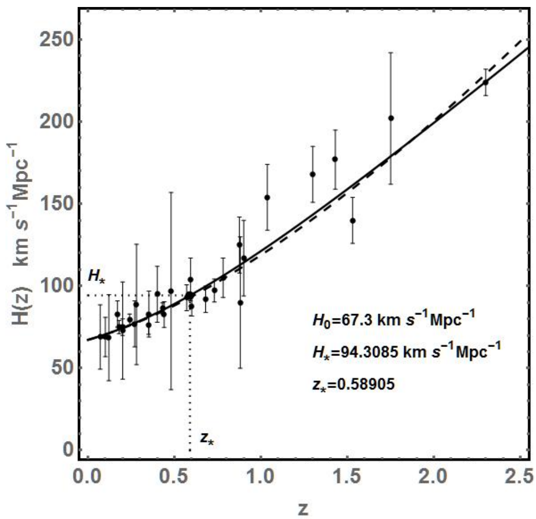

Figure 3. The best fit of the theoretical $H(z)$ dependence to the observational data of Ia type supernovae for the discussed SE-model.

variable t is a one-to-one and increasing function of $\bar{v}$ (see (9)). Therefore, every moment of the discussed time evolution can be ascribed a unique value of the $\bar{\nu}$ variable. The most interesting moments of the time evolution seem to be the present moment t_0 (or $\bar{\nu}_0$), the moment t_* (or $\bar{\nu}_*$) of the commencement of the late acceleration and the final moment $t \to \infty$ (or $\bar{\nu}_f$).

Using the obtained best-fit values of γ_1 and $\bar{\gamma}_2$ and with the help of formulas (12,38,44) and the analysis of the $\bar{a}(\bar{\nu})$ dependence, we can calculate $\bar{\nu}_0 = 6.38094 \times 10^{17}\ s$, $\bar{\nu}_f = 1.43795 \times 10^{18}\ s$, $\bar{\nu}_* = 3.73357 \times 10^{17}\ s$ and $\Lambda_{th} = 1.06803 \times 10^{-52}\,\mathrm{m}^{-2}$. These parameters enable us to obtain values of the following important quantities characteristic for our model i.e. the age of the universe $t_0 = 14.8063 \times 10^9$ y, the acceleration commencement moment $t_* = 8.96978 \times 10^9$ y, the redshift $z_* = z(\bar{\nu}_*) = 0.589055$ and the value of the Hubble function at the acceleration commencement moment $H_* = H(\bar{\nu}_*) = 94.3085 \times \mathrm{km\,s^{-1}Mpc^{-1}}$. The theoretically calculated quantities t_0, t_*, z_*, H_* and $\Lambda := \Lambda_{th}$ are thus in agreement with the observational results.

The fact that the value of Λ_{th} agrees with the results of the Planck Mission favors our ΛSE-model. For this model we can draw a graph of the $w_\Lambda(\bar{\nu})$ dependence (Figure 4). The Figure confirms results of Example 4 and additionally demonstrates that in the deceleration-acceleration period pressure p of matter in such universe was positive. In the next parts of the paper we will concentrate our discussion on the ΛSE-model.

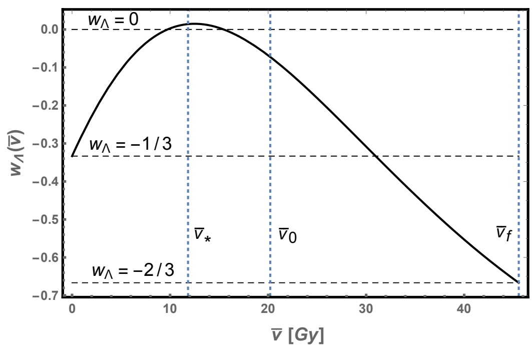

Figure 4. The $w_\Lambda(\bar{\nu})$ dependence for the ΛSE-model best fit to the observational data.

7. Summary

The smoothness equation (3) is a result of a strictly geometrical discussion of the assumption that orientation with respect to the cosmological time makes sense also on the manifold's d-closures for the flat FRW-models Gruszczak 2014. That assumption is a very restrictive condition. The ΛSE-model constitutes the solution of the smoothness equation. It is a very intriguing fact that this strictly geometrical considerations lead to the ΛSE-model which agrees with the observational data (see Section 6).

Our model does not contain fluids usually considered as "ordinary matter". It contains the string gas, two types of domain walls and vacuums: the γ_1-vacuum, the γ_2-vacuum, the γ_2^b-vacuum and the γ_{12}^b-vacuum. The fluids are interacting fluids.

Let us trace the role of fluids discussed in Example 4 in the important moments of the ΛSE-model evolution.

In the first stages of the evolution ($\bar{a} \approx 0$) the dominating forms of matter are the string gas and the γ_1-domain walls. The cause that our ΛSE-model decelerates expansion are the γ_1-domain walls (see Example 2). The remaining fluids have now a little influence on the rate of the expansion.

In the middle stages of the evolution ($\bar{\nu} \approx \bar{\nu}_*$ or $\hat{t} \approx \hat{t}_*$) the moderate influence of the γ_1-domain walls vanishes. Now the dominating role is played by all of the γ_2-fluids and the γ_1-vacuum. The fluids cause the change from deceleration to acceleration (Figures 2 and 4).

At the final stage of the ΛSE-universe evolution, its expansion accelerates to infinity (see Figure 2) and the dominating form of matter are the γ_2^b-domain walls (Examples 3 and 4). In the end the γ_2^b-domain walls disappear due to expansion.

In our next papers we will discuss a SE-model which additionally contains dust and radiation.

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