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Functorial Lie Groups

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Abstract
In this paper we present the concept of a functorial Lie group. We enrich the space with parametric points and consider the structure as C^∞ – algebra [Pysiak et al. 2020]. We get varied spaces. Taking the Lie group as a space, we construct the functorial Lie group with parametric points. We also consider the Lie algebra of the functorial Lie group and construct the functorial exponential map.

Keywords: Lie group, Lie algebra, left – invariant vector field, parametric points, functorial Lie group, exponential map

1. Preliminaries

Let $(G, C^\infty(G))$ be any Lie group, i.e. a group which also has the structure of a manifold, with the group operations:

$$\begin{aligned} \mu : G \times G \to G, \quad \mu(g_1, g_2) = g_1 g_2, \\ \nu : G \to G, \quad \nu(g) = g^{-1}, \end{aligned} \tag{1}$$

for any $g_1, g_2, g \in G$ and the neutral element $e \in G$.

For $a \in G$ we define mappings: the left translation $L_a : G \to G$ in the group G by the element a, $L_a(g) = ag$, the right translation $R_a : G \to G$ in the group G by the element a, $R_a(g) = ga$, and the inner automorphism $\mathrm{ad}_a : G \to G$, $\mathrm{ad}_a(g) = aga^{-1}$.

The mappings R_a and L_a are diffeomorphisms. They induce differential maps (the push-forwards) $(R_a)_{*g} : T_gG \to T_{ga}G$ and $(L_a)_{*g} : T_gG \to T_{ag}G$ respectively, which are given by the formulas:

$$((R_a)_* v)(f) = v(f \circ R_a),$$

$$((L_a)_* v)(f) = v(f \circ L_a)$$

for any $v \in T_gG$ and $f \in C^\infty(G)$.

Let $X : G \to TG$ be a left – invariant vector field on G, i.e. the vector field which is invariant with respect to any left translation on G:

$$\forall a \in G \quad (L_a)_*(X) = X,$$

or equivalently

$$\forall a \in G \quad (L_a)_* X(e) = X(a).$$

Let us denote by $\mathcal{L}(G)$ the set of all left - invariant vector fields on G,

$$\mathcal{L}(G) = \{X \in \mathcal{X}(G) : \forall a \in G \; (L_a)_*(X) = X\}.$$

$\mathcal{L}(G)$ is a Lie algebra, which is called the *Lie algebra of the Lie group* G. It is the set of all left - invariant vector fields on G with the usual addition, scalar multiplication and bracket operation [Kobayashi and Nomizu 1963].

The mapping $i_G : \mathcal{L}(G) \to T_eG$, $i_G(X) = X(e)$, is an isomorphism of linear spaces [Gancarzewicz 2010]. So it allows us to transfer the Lie bracket $[\;,\,]$ from $\mathcal{L}(G)$ to T_eG and

$$\dim \mathcal{L}(G) = \dim T_eG.$$

$\mathcal{L}(G)$ is the Lie subalgebra of the dimension n ($n = \dim G$) of the Lie algebra of all vector fields $\mathcal{X}(G)$. It is known [Kobayashi and Nomizu 1963], that every $X \in \mathcal{L}(G)$ generates a global flow as 1-parameter group of transformations of G, i.e. C^∞ mapping $\varphi : \mathbf{R} \times G \to G$, $\varphi_t : G \to G$, given by $\varphi_t(g) = \varphi(t, g)$, satisfying the conditions:

1. φ_t is diffeomorphism for all $t \in \mathbf{R}$,
2. $\forall_{t,s\in\mathbf{R}} \forall_{g\in G} \quad \varphi_t(\varphi_s(g)) = \varphi_{t+s}(g)$,
3. $\forall_{g\in G} \quad \varphi_0(g) = g$.

For fixed point $g \in G$, $\varphi_g(t)$ is the integral curve of the field X starting at $g \in G$. We have

$$X(\varphi_g(t)) = \frac{d\varphi_g(s)}{ds}\Big|_{s=t} = \varphi_g'(t).$$

Theorem 1.1 *If $X \in \mathcal{L}(G)$ and φ_t is the global flow of X, then for $a_t := \varphi_t(e)$, $t \in \mathbf{R}$, we have the following formulas [Gancarzewicz 2010]:*

1. $a_t \cdot a_s = a_{t+s}$,
2. $a_0 = \varphi_0(e) = e$,
3. $\varphi_t = R_{a_t}$.

The family $\{a_t\}_{t\in\mathbf{R}}$ is called *the 1-parameter subgroup of G generated by an element* $X \in \mathcal{L}(G)$. In another characterisation a_t is a unique curve in G such that its tangent vector $\dot{a}_t$ at a_t is equal to $L_{a_t}(X(e))$. In other words, a_t is a unique solution of the differential equation

$$a_t^{-1}\dot{a}_t = X(e)$$

with initial condition $a_0 = e$.
Theorem 1.1 allows us to define a mapping

$$\exp_G : \mathcal{L}(G) \to G,$$

$$\exp_G(A) = a_1 = \varphi_1(e),$$

which is called the *exponential mapping*.
The exponential mapping $\exp_G : \mathcal{L}(G) \to G$ is smooth and has the following properties:

Theorem 1.2 *If $\{a_t\}_{t\in\mathbf{R}}$ is the 1-parameter subgroup of G generated by an element $A \in \mathcal{L}(G)$, then:*

1. *for any $s \in \mathbf{R}$, $\exp_G(sA) = a_s$,*
2. *for $k \in \mathbf{Z}$, $\exp_G(kA) = (\exp_G(A))^k$,*
3. $\exp_G(0) = a_0 = e$.

2. Space with parametric points

Let $(M, C^\infty(M))$ be a differential manifold, $(P, C^\infty(P))$ be a manifold of parameters. Every mapping $\varphi : P \to M$ is called *the parametric point of the manifold M.* Spaces with parametric points were considered in [Navarro Gonzalez and Salas J. B. 2003], [Pysiak et al. 2020], [Falkiewicz, E. and Sasin, W. 2021].

By $\bar{M}(P)$ we denote the set of all parametric points $\varphi : P \to M$ of the manifold M. Let us notice that

$$\bar{M}(P) = M^P.$$

For any smooth function $f \in C^\infty(M)$ we define $\bar{f} : \bar{M}(P) \to C^\infty(P)$ by the formula

$$\bar{f}(\varphi) = f \circ \varphi \quad \text{for} \quad \varphi \in M^P. \tag{2}$$

By the symbol $C^\infty(\bar{M}(P))$ we denote the set of functions $\bar{f}$, where $f \in C^\infty(M)$,

$$C^\infty(\bar{M}(P)) = \{\bar{f} : f \in C^\infty(M)\}.$$

It is easy to see that $C^\infty(\bar{M}(P))$ is a C^∞ - algebra with operations

$$\omega(\bar{f}_1, \ldots, \bar{f}_n) = \overline{\omega(f_1, \ldots, f_n)}$$

for any $\omega \in C^\infty(\mathbf{R}^n)$, $n \in \mathbf{N}, f_1, \ldots, f_n \in C^\infty(M)$. We obtain the ringed space $(\bar{M}(P), C^\infty(\bar{M}(P)))$ of parametric points with parameters from P, associated with the manifold $(M, C^\infty(M))$.

For any smooth mapping $F : M \to N$ of manifolds M and N we define the mapping

$$\bar{F} : \bar{M}(P) \to \bar{N}(P)$$

by the formula

$$\bar{F}(\varphi) = F \circ \varphi.$$

The mapping $\bar{F}$ is smooth, $\bar{F} : (\bar{M}(P), C^\infty(\bar{M}(P))) \to (\bar{N}(P), C^\infty(\bar{N}(P)))$. Indeed,

$$\bar{F}^*(\bar{\beta}) = \overline{F^*(\beta)} = \overline{\beta \circ F} \quad \text{for} \quad \beta \in C^\infty(N).$$

Lemma 2.1 *The mapping $J : C^\infty(M) \to C^\infty(\bar{M}(P))$ given by*

$$J(f) = \bar{f}$$

is an isomorphism of the C^∞ - rings.

Proof: If $\bar{f} = \bar{g}$ then $\forall \varphi \in M^P$ we have $\bar{f}(\varphi) = \bar{g}(\varphi)$, i.e. $f \circ \varphi = g \circ \varphi$. Then $\forall p \in P$, $f \circ \varphi(p) = g \circ \varphi(p)$ or otherwise $f(\varphi(p)) = g(\varphi(p))$. From this we obtain $f = g$ $\forall x = \varphi(p) \in M$. QED

More generally, we construct a functor

$$\bar{M} : \mathbf{Diff} \to \mathbf{Sets},$$

where **Diff** is a category of differential manifolds and **Sets** is a category of sets. At a stage $P \in \mathbf{Diff}$ we have

$$\bar{M}(P) = M^P.$$

For any function $f \in C^\infty(M)$ we define 1-parameter family of natural transformations as follows:

$$\bar{f} := \{\bar{f}^P\}_{P \in \mathbf{Diff}},$$

At a stage P the mapping $\bar{f}^P : \bar{M}(P) \to C^\infty(P)$ is defined by (2), $\bar{f}^P(\varphi) = f \circ \varphi$.

We obtain the ringed space $(\bar{M}, C^\infty(\bar{M})$ with the C^∞ - algebra $C^\infty(\bar{M})$ isomorphic to the C^∞ - algebra $C^\infty(M)$ (see [Palais 1981], [Sikorski 1972], [Nestruev 2003]).

3. Lie groups with parametric points

Let $(G, C^\infty(G))$ be the Lie group with the actions (1). Let us consider space $(\bar{G}(P), C^\infty(\bar{G}(P)))$ of parametric points $\varphi : P \to G$, associated with the manifold $(G, C^\infty(G))$.
The smooth mappings $\bar{\mu} : \bar{G}(P) \times \bar{G}(P) \to \bar{G}(P)$ and $\bar{\nu} : \bar{G}(P) \to \bar{G}(P)$ are group actions in $\bar{G}(P)$. We obtain *differential Lie group* $(\bar{G}(P), C^\infty(\bar{G}(P)))$ with group actions

$$\varphi_1 \cdot \varphi_2 = \bar{\mu}(\varphi_1, \varphi_2) \text{ and } \varphi^{-1} = \bar{\nu}(\varphi)$$

for any parametric points $\varphi, \varphi_2, \varphi \in \bar{G}(P) = G^P$. The neutral element of $\bar{G}(P)$ is constant mapping

$$\varphi_e : P \to G, \quad \varphi_e(p) = e \text{ for any } p \in P.$$

Obviously

$$\varphi_1 \cdot \varphi_2 = \bar{\mu}(\varphi_1, \varphi_2) = \mu \circ (\varphi_1, \varphi_2) = \mu(\varphi_1, \varphi_2),$$
$$\varphi^{-1} = \bar{\nu}(\varphi) = \nu \circ \varphi.$$

From the above

$$(\varphi_1, \varphi_2)(p) = \varphi_1(p) \cdot \varphi_2(p), \quad \varphi^{-1}(p) = (\varphi(p))^{-1} \text{ for all } p \in P.$$

Proposition 3.1 *$(\bar{G}(P), C^\infty(\bar{G}(P)))$ is the Lie group with parametric points, associated with the Lie group $(G, C^\infty(G))$.*

By Lemma 2.1 $C^\infty(\bar{G}(P)) = \{\bar{f} : \bar{f}(\varphi) = f \circ \varphi, \varphi \in G^P\}$ is a C^∞ – algebra isomorphic to $C^\infty(G), f \mapsto \bar{f}$ is an isomorphism. From this fact we obtain that any derivation $X \in \mathrm{Der}(C^\infty(G))$, $X : C^\infty(G) \to C^\infty(G)$ induces the derivation $\bar{X} \in \mathrm{Der}(C^\infty(\bar{G}))$:

$$\bar{X}\bar{f} = \overline{Xf}.$$

The extension $\bar{X}$ of a derivation X is unique, i.e. if $\bar{X} = \bar{Y}$ then $X = Y$.
If we take the constant mapping $\epsilon : P \to G$, where $\epsilon(p) = e$ for all $p \in P$, then we obtain

$$\bar{X}(\epsilon) = X \circ \epsilon = X(e).$$

We say that the derivation $\bar{X} \in \mathrm{Der}(C^\infty(\bar{G}))$, $\bar{X} : C^\infty(\bar{G}) \to C^\infty(\bar{G})$, is left-invariant, if for any $\varphi \in G^P$,

$$(L_\varphi)_* \bar{X}(\epsilon) = \bar{X}(\varphi). \tag{3}$$

Let us denote by $\mathcal{L}(\bar{G})$ the set of all left-invariant derivations $\bar{X}$ of the algebra $C^\infty(\bar{G})$. $\mathcal{L}(\bar{G})$ is the Lie algebra with the bracket operation $[\cdot, \cdot] : C^\infty(\bar{G}) \times C^\infty(\bar{G}) \to C^\infty(\bar{G})$,

$$[\bar{X}, \bar{Y}] = \bar{X} \circ \bar{Y} - \bar{Y} \circ \bar{X} = \overline{X \circ Y - Y \circ X} = \overline{[X, Y]}.$$

The algebra $\mathcal{L}(\bar{G})$ will be called *the Lie algebra of the functorial Lie group* $\bar{G}$.
On a stage P we have $\mathcal{L}(G^P)$ – the Lie algebra of the Lie group G^P of parametric points.

Proposition 3.2 *Any left-invariant derivation $X \in \mathcal{L}(G)$ induces the left-invariant derivation $\bar{X} \in \mathcal{L}(\bar{G})$.*

Sketch of the proof. Analogously like in a classical case, each left-invariant vector field $\bar{X} \in \mathcal{L}(\bar{G})$ induces its global flow $\bar{\varphi} : \bar{\mathbf{R}} \times \bar{G}(P) \to \bar{G}(P)$ (otherwise $\bar{\varphi} : \mathbf{R}^P \times G^P \to G^P$) which we define by the formula

$$\bar{\varphi}(\tau, \gamma) = \bar{\varphi}_\tau(\gamma)$$

for fixed $\tau : P \to \mathbf{R}$ and for any parametric point $\gamma \in G^P, \gamma : P \to G$. $\bar{\varphi}_\tau$ is the 1-parameter group of transformations of $\bar{G}(P)$ generated by the left – invariant vector field $\bar{X} \in \mathcal{L}(G^P)$.
$\bar{\varphi}_\tau$ possesses similar properties to classical case:

1. $\bar{\varphi}_\tau$ is diffeomorphism for all $\tau \in \mathbf{R}^P$,
2. $\forall_{\tau_1,\tau_2 \in \mathbf{R}^P} \forall_{\gamma \in G^P} \quad \bar{\varphi}_{\tau_1}(\bar{\varphi}_{\tau_2}(\gamma)) = \bar{\varphi}_{\tau_1+\tau_2}(\gamma)$,
3. $\forall_{\gamma \in G^P} \quad \bar{\varphi}_\theta(\gamma) = \gamma$ (or equivalently $\bar{\varphi}_\theta = \mathrm{id}_{G^P}$), where $\theta : P \to \mathbf{R}$, $\forall_{p \in P}\ \theta(p) = 0$.

For fixed $\gamma \in \bar{G}(P)$, $\bar{\varphi}_\gamma(\tau)$ is the integral curve of the vector field $\bar{X}$ starting in $\gamma \in \bar{G}(P)$, where $\tau \in \mathbf{R}^P$. We have

$$\bar{X}(\bar{\varphi}_\gamma(\tau)) = \frac{d\bar{\varphi}_\gamma(\sigma)}{d\sigma}\,|_{\sigma=\tau} = \bar{\varphi}'_\gamma(\tau).$$

Lemma 3.1 *If $\bar{X} \in \mathcal{L}(\bar{G})$ is a left-invariant vector field on $\bar{G}(P)$ corresponding to the left-invariant vector field $X \in \mathcal{L}(G)$ on G via isomorphism of C^∞ - algebras $J : C^\infty(G) \to C^\infty(\bar{G}(P))$ (lemma 2.1), then the global flows for both vector fields are bijective, i.e. for any integral curve of the vector field X there is exactly one integral curve of the vector field $\bar{X}$.*

Taking the left-invariant vector field $\bar{X} \in \mathcal{L}(\bar{G})$ and the global flow $\bar{\varphi}_\tau$ of $\bar{X}$, we define

$$\bar{a}_\tau := \bar{\varphi}_\tau(\epsilon) \in G^P,$$

where $\epsilon \in G^P$ is the constant mapping $\epsilon : P \to G$ such that $\epsilon(p) = e$ for all $p \in P$, where e is the neutral element in the Lie group G. $\bar{a}_\tau$ has properties:

1. $\bar{a}_{\tau_1} \cdot \bar{a}_{\tau_2} = \bar{a}_{\tau_1+\tau_2}$ for $\tau_1, \tau_2 \in \mathbf{R}^P$,
2. $\bar{a}_\theta = \bar{\varphi}_\theta(\epsilon) = \epsilon$, for $\theta \in \mathbf{R}^P, \theta(p) = 0$ for all $p \in P$.

The family $\{\bar{a}_\tau\}_{\tau \in \mathbf{R}^P}$ is called *the 1-parameter subgroup of $\bar{G}(P)$ generated by the left - invariant vector field $\bar{X} \in \mathcal{L}(\bar{G}(P))$*.
Further, using $\bar{a}_\tau$, we define the *exponential mapping for the Lie group $\bar{G}(P)$*:

$$\exp_{\bar{G}(P)} : \mathcal{L}(\bar{G}(P)) \to \bar{G}(P),$$

$$\exp_{\bar{G}(P)}(\bar{X}) = \bar{a}_{\mathbf{1}} = \bar{\varphi}_{\mathbf{1}}(\epsilon),$$

where $\mathbf{1} : P \to \mathbf{R}$ is the constant mapping such that $\mathbf{1}(p) = 1$ for each $p \in P$.

Theorem 3.1 *If $\{\bar{a}_\tau\}_{\tau \in \mathbf{R}^P}$ is the 1-parameter subgroup of $\bar{G}(P)$ generated by the element $\bar{X} \in \mathcal{L}(\bar{G}(P))$ for $\tau \in \mathbf{R}^P$, then:*

1. *for any $\sigma \in \mathbf{R}^P$, $\exp_{\bar{G}(P)}(\sigma \cdot \bar{X}) = \bar{a}_\sigma$ (because $\{\bar{a}_{\sigma\cdot\tau}\}$ is the 1-parameter subgroup of $\bar{G}(P)$ generated by the left - invariant vector field $\sigma \cdot \bar{X} \in \mathcal{L}(\bar{G}(P))$),*
2. *for $k \in \mathbf{Z}$ and $\boldsymbol{k} : P \to \mathbf{R}$ such that $\boldsymbol{k}(p) = k$ for any $p \in P$, $\exp_{\bar{G}(P)}(\boldsymbol{k} \cdot \bar{X}) = \bar{a}_{\boldsymbol{k}} = \bar{a}_{\mathbf{1}} \cdot \ldots \cdot \bar{a}_{\mathbf{1}} = (\exp_{\bar{G}(P)}(\bar{X}))^k$,*
3. *$\exp_{\bar{G}(P)}((\tau_1 + \tau_2)\bar{X}) = \exp_{\bar{G}(P)}(\tau_1\bar{X}) \exp_{\bar{G}(P)}(\tau_2\bar{X})$ for any $\tau_1, \tau_2 \in \mathbf{R}^P$,*
4. *$\exp_{\bar{G}(P)}(\theta) = \bar{a}_\theta = \bar{\varphi}_\theta(\epsilon) = \epsilon$.*

4. Classical Lie groups with parametric points

Let us denote by

$$\mathrm{GL}(n, \mathbf{R}) = \{A \in M_n(\mathbf{R}) : \det A \neq 0\}$$

the well known classical linear Lie group of $n \times n$ real, invertible matrices.

Let $(P, C^\infty(P))$ be a differential manifold of parameters. The associated group to $\mathrm{GL}(n, \mathbf{R})$, with parametric points $\varphi : P \to \mathrm{GL}(n, \mathbf{R})$, is of the form:

$$\mathrm{GL}(n, \mathbf{R}^P) = \mathrm{GL}(n, C^\infty(P)) = \{\varphi : P \to \mathrm{GL}(n, \mathbf{R}) : \forall_{p \in P}\ \varphi(p) \in \mathrm{GL}(n, \mathbf{R})\} =$$

$$= \{\varphi = (\varphi_{ij}) : \varphi_{ij} \in C^\infty(P), i, j = 1, \ldots, n\}.$$

The parametric points of the form $\varphi : P \to \mathrm{GL}(n, \mathbf{R})$ of $\mathrm{GL}(n, C^\infty(P))$, are $n \times n$ invertible matrices, such that $\det \varphi \not\equiv \theta$, where $\theta : P \to \mathbf{R}$ is the constant function satisfying the condition $\theta(p) = 0$ for all $p \in P$.
Analogously to each classical Lie group G of $n \times n$ matrices we can assign the group with parametric points:

$$\bar{G}(P) = G^P = \{\varphi : P \to G : \forall_{p \in P}\ \varphi(p) \in G\} = \{\varphi = (\varphi_{ij}) : \varphi_{ij} \in C^\infty(P), i, j = 1, \ldots, n\}.$$

In other words, for each classical Lie group $G(n, \mathbf{R})$ of $n \times n$ real matrices we can assign functorial group $G(n, \overline{\mathbf{R}})$ of $n \times n$ matrices with elements in $\overline{\mathbf{R}}$, where $\overline{\mathbf{R}}$ is real line with parametric points, $\overline{\mathbf{R}}(P) = \mathbf{R}^P = C^\infty(P)$.

For example, taking classical special orthogonal group $G = \mathrm{SO}(n, \mathbf{R})$, we can construct the functorial group

$$\mathrm{SO}(n, \overline{\mathbf{R}}) = \{A \in M_n(\overline{\mathbf{R}}) : AA^T = A^T A = I \wedge \det A = 1\} = \{A = (a_{ij}) : a_{ij} \in \overline{\mathbf{R}}\}.$$

Let us consider a mapping

$$\bar{\psi} : \mathbf{R}^P \to \mathrm{GL}(n, \mathbf{R}^P),$$

given by the following formula

$$\bar{\psi}(\tau) = \sum_{i=0}^{\infty} \frac{1}{i!}(\tau\varphi)^i = \sum_{i=0}^{\infty} \frac{1}{i!}\tau^i \varphi^i, \tag{4}$$

where $\varphi = (\varphi_{ij}) \in M(n, \mathbf{R}^P)$ is a fixed matrix of functions $\varphi_{ij} : P \to \mathbf{R}$, $\tau \in \mathbf{R}^P$. In the formula (4) $\tau\varphi = \tau \cdot (\varphi_{ij}) = (\tau \cdot \varphi_{ij}) \in M(n, \mathbf{R}^P)$ is a matrix of functions such that for any point $p \in P$ we have

$$(\tau \cdot \varphi_{ij})(p) = (\tau(p) \cdot \varphi_{ij}(p)) \in M(n, \mathbf{R}).$$

We can write the formula (4) in the form

$$\bar{\psi}(\tau) = \mathbf{I} + \tau\varphi + \frac{1}{2}(\tau\varphi)^2 + \frac{1}{3!}(\tau\varphi)^3 + \ldots,$$

where $\mathbf{I} = \begin{pmatrix} \mathbf{1} & & \theta \\ & \ddots & \\ \theta & & \mathbf{1} \end{pmatrix} \in \mathrm{GL}(n, \mathbf{R}^P)$ is a unitary matrix of functions, $\mathbf{1} : P \to \mathbf{R}$ and $\theta : P \to \mathbf{R}$ are the constant mappings such that $\mathbf{1}(p) = 1$ for all $p \in P$ and $\theta(p) = 0$ for all $p \in P$.

The mapping $\bar{\psi}$ defined by the formula (4) has the following properties:

1. $\bar{\psi}(\theta)(p) = I \in \mathrm{GL}(n, \mathbf{R})$ for all $p \in P$,
2. $\bar{\psi}(\tau_1 + \tau_2)(p) = \bar{\psi}(\tau_1)(p)\bar{\psi}(\tau_2)(p)$ for all $p \in P$,
3. $(\frac{d}{d\tau}\bar{\psi}(\tau)\mid_{\tau=\theta})(p) = \varphi(p)$ for all $p \in P$.

We obtain the 1-parameter subgroup of $\mathrm{GL}(n, \mathbf{R}^P)$ generated by an element $\varphi \in M(n, \mathbf{R}^P)$:

$$\bar{a}_\tau = \bar{\psi}(\tau) = \exp_{\bar{G}(P)}(\tau\varphi) \in \mathrm{GL}(n, \mathbf{R}^P).$$

The Lie algebra of linear Lie group $\mathrm{GL}(n, \mathbf{R}^P)$ is the set of all $n \times n$ matrices with elements in $\mathbf{R}^P$, $\mathcal{L}(\mathrm{GL}(n, \mathbf{R}^P)) = M(n, \mathbf{R}^P)$. The exponential mapping is

$$\exp_{\bar{G}(P)} : \mathcal{L}(\mathrm{GL}(n, \mathbf{R}^P)) \to \mathrm{GL}(n, \mathbf{R}^P),$$

$$\exp_{\bar{G}(P)}(\varphi) = \bar{a}_{\mathbf{1}} = \bar{\psi}(\mathbf{1}), \tag{5}$$

where $\mathbf{1} : P \to \mathbf{R}$ is the constant mapping such that $\mathbf{1}(p) = 1$ for all $p \in P$.
Let us note that $\bar{a}_\epsilon(p) \in \mathrm{GL}(n, \mathbf{R})$ for all $p \in P$.

Corollary 4.1 *The exponential map* $\exp_{\bar{G}(P)}$ *on the Lie algebra* $\mathcal{L}(\mathrm{GL}(n, \mathbf{R}^P))$ *can be written by the formula*

$$\exp_{\bar{G}(P)}(\varphi) = \sum_{i=0}^{\infty} \frac{1}{i!}\varphi^i, \tag{6}$$

where the above series is pointwise convergent, i.e. $(\exp_{\bar{G}(P)}(\varphi))(p) = \sum_{i=0}^{\infty} \frac{1}{i!}\varphi^i(p)$ is convergent for all $p \in P$.

5. Conclusion

To cocnclude, the functorial group $\bar{G}$ can be considered as the natural generalisation of the Lie group G. The Grothendieck functor $G \mapsto \bar{G}$ infers an isomorphism of the C^∞ - algebra $C^\infty(G)$ of the Lie group G and the C^∞ - algebra $C^\infty(\bar{G}(P))$ of functorial group $\bar{G}(P)$ on a stage P. The basic concepts of the Lie group G have equivalents in the functorial group $\bar{G}(P)$ at a stage P and in the functorial group $\bar{G}$ at any stage. Each concept on the C^∞ - algebra $C^\infty(G)$ is associated with the corresponding concept of the C^∞ - algebra $C^\infty(\bar{G}(P))$. Despite the isomorphism of the C^∞ - algebras $C^\infty(G)$ and $C^\infty(\bar{G}(P))$, the differential groups $(G, C^\infty(G))$ and $(\bar{G}(P), C^\infty(\bar{G}(P)))$ are not isomorphic as ordered pairs, because sets of points G and $\bar{G}(P)$ are not bijective.

The study of the properties of functorial groups will be continued, owing to the substantial role of these symmetry groups and the range of their applications, particularly in the theory of functorial principal bundles associated with the corresponding classical principal bundles.

References

Falkiewicz, E. and Sasin, W. 2021. Zastosowania Matematyki w Aktuyalnych Problemach Technicznych, Ekonomicznych i Społecznych. Chap. Twierdzenie Weila w funktorialnym module różniczkowym, edited by Marek Wójtowicz. Radom, Poland: Wydawnictwo Uniwersytetu Technologiczno - Humanistycznego im. Kazimierza Pułaskiego w Radomiu.

Kobayashi, S., and K. Nomizu. 1963. *Foundations of Differential Geometry, Vol. I.* New York, London: Interscience Publishers.

Navarro Gonzalez, J. A., and Sancho de Salas J. B. 2003. *C^∞ - Differentiable Spaces.* Berlin, Heidelberg, New York: Springer Verlag.

Nestruev, J. 2003. *Smooth Manifolds and Observables.* Berlin, Heidelberg, New York: Springer Verlag.

Palais, R. S. 1981. *Real Algebraic Differential Topology, Part I.* Brandeis University: Publish or Perish, Inc. https://www.researchgate.net/publication/259714095_Real_Algebraic_Differential_Topology.

Pysiak, L., W. Sasin, M. Heller, and T. Miller. 2020. Functorial differential spaces and the infinitesimal structure of space-time. *Reports on Mathematical Physics* 85(3):443–454.

Gancarzewicz, J. 2010. *Zarys współczesnej geometrii różniczkowej.* Script. Warszawa.

Sikorski, R. 1972. *Introduction to Differential Geometry.* In Polish. Warszawa: PWN.