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Invitation to Functorial Spaces

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Abstract

We propose a generalization of the differential space concept that we call functorial differential space. It consists in replacing the standard differential structure $C^\infty(M)$ of a differential space by the differential structure consisting of functions from an extended space $\bar{M}(A)$ to an algebra A, satisfying certain general conditions, but not necessarily even commutative, and $\bar{M}(A)$ being a set of mappings $C^\infty(M) \to A$, interprtetd as new points. In this way, we obtain a whole family of theories of differential spaces (depending on the algebra A). It is this family that is a functorial differential space. An important class of spaces arises if we tensor multiply the differential structure of a functorial differential space by a Grassmann algebra. This leads to the concept of a functorial differential superspace (or supermanifold). In this context we also consider Einstein-Grassmann functorial super algebras.

Keywords: differential structure, functorial geometry, supermanifolds, Einstein-Grassmann algebra

1. Introduction

In papers (Pysiak et al. 2020) and (Pysiak et al. 2023), we used a method, called by us the functorial method, to study the infinitesimal structure of space-time and Einstein algebras. Broadly speaking, it consists in replacing, in the theory of differential spaces, the differential structure, consisting of smooth functions from a set M of points to the space of real numbers $\mathbb{R}$, i.e. functions of $C^\infty(M)$, with a structure consisting of smooth functions from an extended space $\bar{M}(A)$ to an algebra A; the latter is supposed to satisfy certain general conditions (smoothness condition could be relaxed); $\bar{M}(A)$ is here a set of mappings $C^\infty(M) \to A$ interpreted as a set of new points. The formalism is constructed in such a way that the algebra A plays the role of a stage from category theory, and the whole becomes a functor from the category of algebras of a certain type to the category of sets. Substituting various algebras for A, we obtain a family of theories of differential spaces. An important class of spaces arises if we tensor multiply the differential structure of a functorial differential space by a Grassmann algebra. This leads to the functorial superspace concept.

Regardless of its applications, the method of functorial differential spaces seems interesting in itself. In this paper, we will make a more systematic review of this method, as it has been developed so far, and refine some of its details.

J. A. Navarro Gonsàlaz and G. J. B. de Silas (Navarro Goncàlez and Sancho de Salas 2003), and Jet Nestruev (Nestruev 2002) should be considered the precursors and at least partly the inspirers of our approach to functorial spaces. In (Navarro Goncàlez and Sancho de Salas 2003), the authors introduced the concept of parametric points as mappings from one differential space to another; in (Nestruev 2002) the author considers the real spectrum of a certain geometric algebra A (that is, an algebra that can be represented as an algebra of real functions on a certain space) as a set of maps from A to $\mathbb{R}$. We should also mention R. S. Palais (Palais 1981), A. Kock (Kock 2006, 2009), and I.

Moerdijk and G. E. Reyes (Moerdijk and Reyes 2010) from whom we have borrowed and developed some ideas and methods.

The plan of our work runs as follows:

- In Section 2, we define the main environment of our analyses, i.e., C^∞-algebras.
- In Section 3, we consider a differential space $(M, C^\infty(M))$ and make a key move by defining a space whose points are maps from from M to an algebra A, and a differential structure on this space. The result of this procedure is a functorial ringed space whose geometric properties we investigate.
- In Section 4, we deal with the issue of generating differential structures for functorial differential spaces.
- In Section 5, we further develop the differential geometry for functorial spaces.
- In Section 6, we define functorial Einstein algebras.
- In Section 7, we construct sheaf version of functorial differential spaces.
- In Section 8, we show that if the differential structure of a functorial differential space is tensor multiplied by a Grassmann algebra, one obtains a functorial super differential space. Correspondingly, one can also speak of Einstein-Grassmann superalgebras.

2. C^∞-algebras

In this section, we recall the definition and some properties of the C^∞-algebra, which sets the context for our further constructions.

Definition 1 *A unital commutative $\mathbb{R}$-algebra A is a C^∞-algebra if, for any $n \in \mathbb{N}$, $\omega \in C^\infty(\mathbb{R}^n)$ and $a_1, \ldots, a_n \in A$, the element $\omega(a_1, \ldots, a_n)$ is defined and the following conditions are satisfied*

1. *for* $\varphi, \psi \in C^\infty(\mathbb{R}^2)$, $a_1 + a_2 = \varphi(a_1, a_2)$, *where* $\varphi(x_1, x_2) = x_1 + x_2$.
   $a_1 \cdot a_2 = \psi(a_1, a_2)$, *where* $\psi(x_1, x_2) = x_1 \cdot x_2$.
2. *for* $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$, $\pi_i(x_1, \cdots, x_n) = x_i$, $i = 1, 2, \cdots, n, a_1, \cdots, a_n \in A$,
$$\pi_i(a_1, \cdots, a_n) = a_i,$$
3. *for the function* $1 \in C^\infty(\mathbb{R}^n)$, $1(x_1, \cdots, x_n) = 1$, $a_1, \cdots, a_n \in A$,
$$1(a_1, \cdots, a_n) = 1_A,$$
4. *for* $\theta \in C^\infty(\mathbb{R}^m)$, $\omega_1, \ldots, \omega_m \in C^\infty(\mathbb{R}^m)$, $a_1, \ldots, a_n \in A$, $m, n \in \mathbb{N}$,
$$(\theta \circ (\omega_1, \ldots, \omega_m))(a_1, \ldots, a_n) = \theta(\omega_1(a_1, \ldots, a_n), \ldots, \omega_m(a_1, \ldots, a_n)).$$

Let A, B be C^∞-algebras. A homomorphism $f : A \to B$ of $\mathbb{R}$-algebras is said to be C^∞-morphism if
$$f(\omega(a_1, \ldots, a_n)) = \omega(f(a_1), \ldots, f(a_n)).$$
C^∞- algebras as objects and C^∞-morphisms as morphisms form a category, denotedby C^∞.

Let us notice that $C^\infty(M)$ is an C^∞-algebra.

3. Ringed Space $(\bar{M}_L(A), C^\infty(\bar{M}_L(A))$

In this section, we make a key move for our subsequent considerations. If $(M, C^\infty(M))$ is a differential manifold (or a differential space), and A some algebra, we define a space whose points are maps from M to A, and a differential structure on this space (which turns out to be a C^∞-algebra). It turns out that this structure is isomorphic to the initial differential structure $C^\infty(M)$. Thanks to this, we can use it not only to build the standard geometry of M, but also by substituting various other algebras for A (treating them as stages in the appropriate category), we can construct completely new geometries.

Let $(M, C^\infty(M))$ be an n-dimensional differential manifold (or a differential space), and A an m-dimensional linear space such that $\mathbb{R} \subset A$. By $\bar{M}_L(A) := [C^\infty(M), A]_L$ we will denote the set of linear maps

$$\rho : C^\infty(M) \to A.$$

$\bar{M}_L(A)$ can be interpreted as a functor from the category of C^∞-algebras to the category of linear spaces.

For every smooth function $f \in C^\infty(M)$ we define the function $\bar{f} : \bar{M}_L(A) \to A$ by

$$\bar{f}(\rho) = \rho(f)$$

for any $\rho \in \bar{M}_L(A)$.

It is easy to see that the evaluation $ev_p : C^\infty(M) \to \mathbb{R}$, $ev_p(f) = f(p)$, for any $p \in M$, belongs to $\bar{M}_L(A)$. The set of all functions $\bar{f}$, for $f \in C^\infty(M)$, will be denoted by $C^\infty(\bar{M}_L(A)) = \{\bar{f} : f \in C^\infty(M)\}$.

Lemma 1 *The mapping $J : C^\infty(M) \to C^\infty(\bar{M}_L(A))$ given by*

$$J(f) = \bar{f}$$

is a bijection.

Proof. Suppose $\bar{f} = \bar{g}$ for some $f, g \in C^\infty(M)$. This implies that $\bar{f}(ev_p) = \bar{g}(ev_p)$ for any $p \in M$. Therefore, $f(p) = g(p)$. And finally, $f = g$. $\square$.

Theorem 1 *$C^\infty(\bar{M}_L(A))$ is a C^∞-algebra with the operation of compositions with C^∞-functions*

$$\omega(\bar{f}_1, \ldots, \bar{f}_n) = \overline{\omega(f_1, \ldots, f_n)}$$

for $\omega \in C^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}, f_1, \ldots, f_n \in C^\infty(M)$.

Proof. We check conditions of the definition.

1. It is easy to see that the addition and multiplicaion are of the form

$$\bar{f}_1 + \bar{f}_2 = \varphi(f_1, f_2)$$

   for $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x_1, x_2) = x_1 + x_2$, and

$$\bar{f}_1 \cdot \bar{f}_2 = \overline{\psi(f_1, f_2)}$$

   for $\psi : \mathbb{R}^2 \to \mathbb{R}$, $\psi(x_1, x_2) = x_1 \cdot x_2$.
2. There is compatibility of C^∞-algebraic operations with the projections: for $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $\pi_i(x_1, \ldots, x_n) = x_i$, $i = 1, \ldots, n$, $x_1, \ldots, x_n \in \mathbb{R}$, we have

$$\pi_i(\bar{f}_1, \ldots, \bar{f}_n) = \overline{\pi_i(f_1, \ldots, f_n)} = \bar{f}_i,$$

   $\bar{f}_1, \ldots, \bar{f}_n \in C^\infty(\bar{M}_L(A))$.
3. For $1 \in C^\infty(\mathbb{R}^n)$, $\bar{f}_1, \ldots, \bar{f}_n \in C^\infty(\bar{M}_L(A))$ we have

$$1(\bar{f}_1, \ldots, \bar{f}_n) = \overline{1(f_1, \ldots, f_n)} = \bar{1}_M \cdot \bar{f} = \overline{1_M \cdot f} = \bar{f}.$$

4. For $\theta \in C^\infty(\mathbb{R}^n)$, $\omega_1, \ldots, \omega_m \in C^\infty(\mathbb{R}^m)$ we have

$$\Theta \circ (\omega_1, \ldots, \omega_m)(\bar{f}_1, \ldots, \bar{f}_n) = \theta(\omega_1(\bar{f}_1, \ldots, \bar{f}_n), \ldots, \omega_m(\bar{f}_1, \ldots, \bar{f}_n)). \ \square$$

Let us notice that addition as define above is a pointwise operation. Indeed,

$$(\bar{f}_1 + \bar{f}_2)(\rho) = (\overline{f_1 + f_2})(\rho) = \rho(f_1 + f_2) = \rho(f_1) + \rho(f_2) = \bar{f}_1(\rho) + \bar{f}_2(\rho)$$

for every $\rho \in \bar{M}_L(A)$. However, multiplication as defined above is not a pointwise operation. Indeed,

$$(\bar{f}_1 \cdot \bar{f}_2)(\rho) = (\overline{f_1 f_2})(\rho) = \rho(f_1 f_2).$$

If ρ is linear but not multiplicative, i.e. if $(f_1 f_2)(\rho) = \rho(f_1)\rho(f_2)$ does not hold, we cannot continue transforming the right-hand side of the above equality. The multiplication operation is pointwise only if ρ is multiplicative, i.e. when we change from the category of linear spaces to the category of C^∞-algebras. In such a case,

$$(\bar{f}_1 \cdot \bar{f}_2)(\rho) = \rho(f_1 f_2) = \rho(f_1)\rho(f_2) = \bar{f}_1(\rho) \cdot \bar{f}_2(\rho).$$

In this way, we obtain the ringed space $(\bar{M}_L(A), C^\infty(\bar{M}_L(A))$ which can be considered as a generalized differential space (in which case we will decorate M with the subscript L). If A is a C^∞-algebra, the linear mappings $\rho : C^\infty(M) \to A$ have the multiplicativity property (in which case we will drop the subscript L). In this case, the corresponding differential space $(\bar{M}(A), C^\infty(\bar{M}(A))$ is a differential subspace of $(\bar{M}_L(A), C^\infty(\bar{M}_L(A))$.

4. Generating Differential Structure of a Functorial Space

In this section, we deal with the issue of generating differential structures for functorial differential spaces. The issue is not obvious because in the case of a functorial space its structure depends on an algebra A. However, we will show that, despite this, the usual method of generating differential structure is nicely transferred to functorial spaces.

Let us recall that the structure of a differential space (M, C) is generated by a subset $C_0 \subset \mathbb{R}^M$ if $C = (scC_0)_M$, and on M we have the topology τ_{C_0}, the weakest topology on M in which the functions from C_0 are continuous. The following Lemma says how to transfer this definition to functorial spaces.

Lemma 2 *If C is generated by C_0 then $\bar{C}(A)$ is generated by $\bar{C}_0(A) = \{\bar{f}^A : f \in C_0\}$.*

Proof. We will show this Lemma locally, for any $U \in \tau_{C_0}$, and for any A (τ_{C_0} is here the weakest topology in which functions of C are continuous), We have

$$\forall_{\rho \in \bar{U}(A)}\, \rho(f|U) = \rho(\omega(g_1, \ldots, g_n)|U \Rightarrow \forall_{\rho \in \bar{U}(A)} \bar{f}(\rho) = \omega(\bar{g}_1, \ldots, \bar{g}_n)(\rho).$$

Therefore, $\bar{f}|\bar{U}(A) = \omega(\bar{g}_1, \ldots, \bar{g}_n)|\bar{U}(A)$. $\square$

We can see that if we have a ringed space $(\bar{M}, \bar{C})$ then $\bar{C}$ is generated by $\bar{C}_0$ if

- $sc\bar{C}_0 = \bar{C}$,
- $(\bar{C}_0)_M = \bar{C}$ (in the sense of sheaf).

Analogous results can be obtained when we allow for linear transformations, i.e. when we consider a ringed space $(\bar{M}_L(A), \bar{C}_L(A))$ with $\bar{M}_L(A) = [C, A]_L$ and $\bar{C}_L(A) = \{\bar{f}_L : f \in C,\ \bar{f}_L(\rho) = \rho(f)$ for $\rho \in \bar{M}_L(A)\}$.

5. Functorial Geometry

In this section, we construct differential geometry for the space $(\bar{M}_L(A), C^\infty(\bar{M}_L(A)))$. From Lemma 1 we know that the C^∞-algebras $C^\infty(M)$ and $C^\infty(\bar{M}_L(A))$ are isomorphic. Therefore, to every derivation $X \in \mathrm{Der}(C^\infty(M))$, there uniquely corresponds the derivation $\bar{X} \in \mathrm{Der}(C^\infty(\bar{M}_L(A)))$ given by $\bar{X}\bar{f} = \overline{Xf}$ for $f \in C^\infty(M)$.

To every metric tensor g on M there uniquely corresponds the metric tensor $\bar{g} : \mathrm{Der}(C^\infty(\bar{M}_L(A))) \times \mathrm{Der}(C^\infty(\bar{M}_L(A))) \to C^\infty(\bar{M}_L(A))$ given by $\bar{g}(\bar{X}, \bar{Y}) = \overline{g(X, Y)}$ for $X, Y \in \mathrm{Der}(C^\infty(M))$.

To every Levi-Civita connection ∇ of a metric tensor g on M there uniquely corresponds the connection $\bar{\nabla} : \mathrm{Der}(C^\infty(\bar{M}_L(A))) \times \mathrm{Der}(C^\infty(\bar{M}_L(A))) \to \mathrm{Der}(C^\infty(\bar{M}_L(A)))$ given by $\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y}$ for $\bar{X}, \bar{Y} \in \mathrm{Der}(C^\infty(\bar{M}_L(A)))$. It satisfies the following conditions

1. $\bar{Z}(\bar{g}(\bar{X}, \bar{Y})) = \bar{g}(\bar{\nabla}_{\bar{Z}}\bar{X}, \bar{Y}) + \bar{g}(\bar{X}, \bar{\nabla}_{\bar{Z}}\bar{Y})$,
2. $[\bar{X}, \bar{Y}] = \bar{\nabla}_{\bar{X}}, \bar{Y} - \bar{\nabla}_{\bar{Y}}, \bar{X}$

for all $\bar{X}, \bar{Y}, \bar{Z} \in \mathrm{Der}(C^\infty(\bar{M}_L(A)))$.

To every curvature tensor $R(X, Y)Z$ of connection ∇ there uniquely corresponds curvature tensor $\bar{R}(\bar{X}, \bar{Y})\bar{Z}$ for connection $\bar{\nabla}$ such that $\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \overline{R(X, Y)Z}$ for any $\bar{X}, \bar{Y}, \bar{Z} \in \mathrm{Der}(C^\infty(\bar{M}_L(A)))$. The following relationships are satisfied

1. $\bar{R}(\bar{X}, \bar{Y})\bar{Z} = -\bar{R}(\bar{Y}, \bar{X})\bar{Z}$,
2. $\bar{g}(\bar{R}(\bar{X}, \bar{Y})\bar{Z}, \bar{U}) = -\bar{g}(\bar{R}(\bar{X}, \bar{Y})\bar{U}, \bar{Z})$,
3. $\bar{R}(\bar{X}, \bar{Y})\bar{Z} + \bar{R}(\bar{Y}, \bar{Z})\bar{X} + \bar{R}(\bar{Z}, \bar{X})\bar{Y} = 0$,
4. $\bar{g}(\bar{R}(\bar{X}, \bar{Y})\bar{Z}, \bar{W}) = \bar{g}(\bar{R}(\bar{Z}, \bar{W})\bar{X}, \bar{Y})$.

In the local basis $\frac{\bar{\partial}}{\partial x^i}$, $i = 1, \ldots, n$, the curvature tensor $\bar{R}$ has the form

$$\bar{R}(\frac{\bar{\partial}}{\partial x^i}, \frac{\bar{\partial}}{\partial x^j})\frac{\bar{\partial}}{\partial x^k} = \sum_{l-1}^{n} \bar{R}^l_{ijk}\frac{\bar{\partial}}{\partial x^l}.$$

The trace of a tensor $\bar{B}$ of type $(1, 1)$

$$\bar{B} : \mathrm{Der}(C^\infty(\bar{M}_L(A))) \to \mathrm{Der}(C^\infty(\bar{M}_L(A)))$$

has locally the form

$$\mathrm{tr}\bar{B} = \mathrm{tr}(\bar{B}^j_i)$$

where $\bar{B}(\frac{\bar{\partial}}{\partial x^i}) = \sum_{j=1}^n \bar{B}^j_i \frac{\bar{\partial}}{\partial x^j}$. Obviously, one has $\mathrm{tr}\bar{B} = \overline{\mathrm{tr}B}$.

Let us consider two tensors $R(X, Y)$, $X, Y \in \mathrm{Der}(C^\infty(M)$ and a tensor $\bar{R}(\bar{X}, \bar{Y})$, $\bar{X}, \bar{Y} \in \mathrm{Der}(C^\infty(\bar{M}_L(A))$ remaining in one-to-one correspondence. One has $\bar{R}_{\bar{X},\bar{Y}} = \overline{R_{X,Y}}$.

As well known, Ricci tensor satisfies the following relation $\mathrm{Ric}(X, Y) = \mathrm{tr}R_{X,Y}$ and correspondingly $\overline{\mathrm{Ric}}(\bar{X}, \bar{Y}) = \mathrm{tr}\bar{R}(\bar{X}, \bar{Y}) = \overline{\mathrm{tr}R_{X,Y}}$.

We know that there exists an operator $\mathcal{R} : \mathrm{Der}(C^\infty(M)) \to \mathrm{Der}(C^\infty(M))$ such that $\mathbf{R}ic(X, Y) = g(\mathcal{R}X, Y)$ where g is a Lorentz metric on $\mathrm{Der}(C^\infty(M))$. Correspondingly, we have an operator $\bar{\mathcal{R}} : \mathrm{Der}(C^\infty(\bar{M}_L(A)) \to \mathrm{Der}(C^\infty(\bar{M}_L(A))$ such that $\mathbf{R}ic(\bar{X}, \bar{Y}) = \bar{g}(\bar{\mathcal{R}}\bar{X}, \bar{Y}) = \overline{g(\mathcal{R}X, Y)}$ where $\bar{g}$ is a Lorentz metric on $\mathrm{Der}(C^\infty(\bar{M}_L(A))$. Then the scalar (or Ricci) curvature is defined as $\bar{r} = \mathrm{tr}\bar{\mathcal{R}}$.

To have a clear picture of the above, let us notice that our starting manifold (or a differential space) $(M, C^\infty(M))$ is embedded into $(\bar{M}_L(A), C^\infty(\bar{M}_L(A))$. Indeed,

$$(M, C^\infty(M)) \cong ((\bar{M}_{ev}(A), C^\infty(\bar{M}_{ev}(A))) \subset (\bar{M}_L(A), C^\infty(\bar{M}_L(A))).$$

The space $\bar{M}_L(A)$ contains $\bar{M}_{ev}(A)$ as a space of linear functionals $\rho : C^\infty(M) \to A$. In particular, evaluations $ev_p : C^\infty(M) \to \mathbb{R} \subset A$ can be interpreted as original points of the functional space $\bar{M}_L(A)$.

As we can see, we already have all the quantities necessary to formulate the functorial Einstein algebra.

6. Functorial Einstein Algebra

The idea of Einstein algebra comes from Geroch (Geroch 1972); it was later developed in (Heller 1992) and (Heller and Sasin 1995). In (Heller et al. 2024), we defined the Lorentz module on a differential manifold (or a differential space) (M, g) as a triple $(\mathrm{Der}(C^\infty(M)), C^\infty(M), g)$, where $\mathrm{Der}C^\infty(M)$ is a $C^\infty(M)$-module of derivations of the algebra $C^\infty(M)$, and the Einstein algebra on (M, g) as a Lorentz module on (M, g) on which the Einstein equations are defined (although there the definition was given a slightly different wording). The functorial version of this definition has the following form

Definition 2 • *Functorial Lorentz module on a functorial Lorentz manifold* $(\bar{M}_L(A), \bar{g})$ *is a triple* $(\mathrm{Der}(C^\infty(\bar{M}_L(A))), C^\infty(\bar{M}_L(A)), \bar{g})$, *where* $\mathrm{Der}(C^\infty(M))$ *is a* $C^\infty(M)$*-module of derivations of the algebra* $C^\infty(\bar{M}_L(A))$.
- *Einstein algebra on a functorial Lorentz manifold* $(\bar{M}_L(A), \bar{g})$ *is a functorial Lorentz module* $(\mathrm{Der}(C^\infty(\bar{M}_L(A))), C^\infty(\bar{M}_L(A)), \bar{g})$ *on which functorial Einstein equations are defined. They are of the form*

$$(i)\ \ \overline{\mathbf{Ric}} = \Lambda\bar{g},$$

$$(ii)\ \ \overline{\mathbf{Ein}} + \Lambda\bar{g} = 8\pi\bar{T}$$

where $\overline{\mathbf{Ein}} = \overline{\mathbf{Ric}} - \frac{1}{2}\bar{r}\bar{g}$ *is called the Enstein tensor,* Λ *is the cosmological constant, and* $\bar{T}$ *is a suitable energy-momentum tensor.*

Sometimes, unless there is danger of confusion, only the expression $C^\infty(\bar{M}(A))$ will be called Einstein algebra. The above functorial (generalized) Einstein equations select from the $C^\infty(\bar{M}_L(A))$-module of derivations $\mathrm{Der}(C^\infty((\bar{M}_L(A))))$ a submodule of derivations satisfying these equations. The Lorentz metric $\bar{g}$ is defined on this submodule. In ordinary general relativity, the dependence on the derivation submodule remains invisible because the full derivation module is assumed to be considered from the very beginning. In this case, Einstein's equations determine only the Lorentz metric.

Let us notice that if $A = \mathbb{R}$, we have $\rho = ev_p : C^\infty(M) \to \mathbb{R}$ given by $ev_p(f) = f(p)$ and the functorial differential space becomes the usual differential space $(M, C^\infty(M))$. The fact that assuming A different from $\mathbb{R}$ may be interesting is demonstrated by an example: The assumption that A is a Grassmann algebra leads to the theory of supermanifolds (see Section 8).

7. Functorial Structured Spaces

According to the method adopted in the previous chapters of these notes, after developing the differential version of the theory of given spaces, one should move to its structured version. In the case of functorial spaces, this transition is natural. The structured counterpart of functorial spaces is the sheaf of differential structures of functorial spaces over the topological space $(M, \mathrm{top}M)$.

We construct such a sheaf in the following way.

1. With every open subset $U \in \mathrm{top}M$ we associate a functorial space $(\bar{U}_L, C^\infty(\bar{U}_L))$.
2. For any two open subsets $U, V \in \mathrm{top}M$ such that $U \subset V$, the restriction operator is defined $\rho^V_U : C^\infty(\bar{V}_L) \to C^\infty(\bar{U}_L)$ by $\rho^V_U(\bar{f}_L) = \bar{f}|U, \bar{f}_L \in C^\infty(\bar{V})$ for $f \in C^\infty(M)$.

Definition 3 *The functorial structured space over a topological space* $(M, \mathrm{top}M)$ *is a pair* $(M, \mathcal{O}_M^L(A))$ *where* $\mathcal{O}_M^L(A)$ *is a sheaf over* $(M, \mathrm{top}M)$ *such that for every* $U \in \mathrm{top}M$ *one has* $\mathcal{O}_M^L(A)(U) = C^\infty(\bar{U}_L(A))$.

Obviously, if A is a C^∞-algebra, the points ρ enjoy the multiplicativity property and the subscript L can be omitted everywhere.

8. Example: Functorial Approach to Supermanifolds

In this section, we show that if the differential structure of a functorial manifold (differential space) is tensor multiplied by a Grassmann algebra, one obtains a functorial supermanifold (functorial super differential space). A Grassmann algebra is an associative $\mathbb{R}$-algebra Λ_n with unity, such that there exists a set of generators in it $1, \beta_1, \ldots, \beta_n$, $n \in \mathbb{N}$, such that

$$1_{\Lambda_n}\beta_i = \beta_i = \beta_i 1_{\Lambda_n}$$

and

$$\beta_i\beta_k + \beta_k\beta_i = 0.$$

Every element $\alpha \in \Lambda_n$ can be written as

$$\alpha = \sum_{k\geq 0}\sum_{i_1,\ldots,i_k} \alpha_{i_1,\ldots,i_k}\beta_{i_1}\ldots\beta_{i_k},$$

Indices $i_1, \ldots, i_k$ are supposed to satisfy one of the following conditions: either $\alpha_{i_1,\ldots,i_k}$ should be antisymmetric with respect to their indices, or the indices should form the increasing sequence $i_1 < i_2 < \ldots < i_k$. Obviously, β_i are nilpotents ($\beta_i^2 = 0$). The above definition of Grassmann algebra can be rephrased in such a way as to allow an infinite number of generators, but we will not deal here with this case.

A unique algebra homomorphism $b : \Lambda_n \to \mathbb{R}$ which maps 1 onto 1 and all the generators to zero is called the body map and its image the body of Λ_n. The linear map s which maps Λ_n onto its nilpotent elements is called the soul map and its image the soul of Λ_n.

Any Grassmann algebra admits a gradation $\Lambda_n = {}^0\Lambda_n \oplus {}^1\Lambda_n$ where ${}^0\Lambda_n$ and ${}^1\Lambda_n$ are spanned by the products of even and odd numbers of generators, respectively. The subspace ${}^0\Lambda_n$ is also a subalgebra of Λ_n. (For more on Grassmann algebras see (Berezin 1983; Berezin and Leites 1975; DeWitt 1984; Rogers 2007).)

There are at least several (not all equivalent) definitions of a supermanifold: for instance, Berezin-Leites original definition (Berezin 1983; Berezin and Leites 1975), Kostant's definition (Kostant 1977), DeWitt's definition in terms of maps and atlases (DeWitt 1984). In what follows, we will refer to Berezin's definition (as it was formulated by Rogers (Rogers 2007, p. 86) for its simplicity. The definition runs as follows.

Definition 4 *A smooth real (algebro-geometric) supermanifold of dimension* (m, n) *is a pair* $(M, \mathcal{A})$ *where* M *is a real* m*-dimensional manifold and* $\mathcal{A}$ *a sheaf of supercommutative algebras over* M *satisfying the following conditions*

1. *there exists an open cover* $\{U_i | U_i \in \mathrm{top}M, i \in I\}$ *of* M *such that for each* $i \in I$

$$\mathcal{A}(U_i) \cong C^\infty(U_i) \otimes \Lambda_n,$$

where Λ_n *is a Grassmann algebra with* n *generators,*

2. *if* $\mathcal{N}$ *is the sheaf of nilpotents in* $\mathcal{A}$*, then*

$$(M, \mathcal{A}/\mathcal{N}) \cong (M, C_M^\infty)$$

where C_M^∞ *is a sheaf of smooth functions on* M.

We will call a supermanifold defined in this way the Berezin supermanifold. Let us notice that an $(m, 0)$-dimensional supermanifold is an m-dimensional Berezin manifold.

Let us now consider a functorial differential space with the differential structure tensorially multiplied by a Grassmann algebra, i.e. $(\bar{M}(A), C^\infty(\bar{M}(A) \otimes \Lambda_n))$, which we change, in the standard way, into the sheaf space $(\bar{M}(A), \mathcal{A}_{\bar{M}})$ with $\mathcal{A}_{\bar{M}}(\bar{U}_i) \cong C^\infty(\bar{U}_i) \otimes \Lambda_n$. We will consider this construction in the *Lin* or C^∞ categories, as appropriate.

Let us first notice that multiplication by a tensor does not spoil the smoothness. The mapping

$$\bar{F} : (\bar{M}(A), C^\infty(\bar{M}(A))) \to (\bar{N}(A), C^\infty(\bar{N}(A))),$$

given by $\bar{F}(\rho) = \rho \circ F^*$, $\rho \in \bar{M}(A)$, is smooth, if for every $\beta \in C^\infty(N)$, one has $\bar{\beta} \circ \bar{F} \in C^\infty(\bar{M}(A))$. It can be easily checked that in our case

$$\bar{F} : (\bar{M}(A), C^\infty(\bar{M}(A)) \otimes \Lambda_n) \to (\bar{N}(A), C^\infty(\bar{N}(A)) \otimes \Lambda_n),$$

this is satisfied. Indeed,

$$(\bar{\beta} \otimes \lambda) \circ \bar{F} = \overline{\beta \circ F} \otimes \Lambda_n,$$

$\lambda \in A$.

The tensor multiplication of the differential structure $C^\infty(\bar{M}_L(A))$ by A has one more consequence: it breaks the isomorphism between the $C^\infty(M)$ and $C^\infty(\bar{M}_L(A))$ structures, thus enriching the geometry of the latter.

Now, let us consider a smooth supermanifold of dimension (m, n) as a pair $(M, \mathcal{A})$ where M is a real m-dimensional manifold and $\mathcal{A}$ a sheaf of supercommutative algebras as in Definition 4. On the strength of the isomorphism

$$J : C^\infty(\bar{U}(A)) \otimes \Lambda_n \to C^\infty(U) \otimes \Lambda_n$$

given by

$$J(\bar{f} \otimes \lambda) = f \otimes \lambda,$$

$\lambda \in A$, we can associate (uniquely) with each Berezin supermanifold $(M, \mathcal{A})$, where, for each $i \in I$, $\mathcal{A}(U_i) \cong C^\infty(U_i) \otimes \Lambda_n$, the functorial space $(\bar{M}(A), \bar{\mathcal{A}})$, where, for each $i \in I$, $\bar{\mathcal{A}}(\bar{U}_i) \cong C^\infty(\bar{U}_i) \otimes \Lambda_n$ (the second condition of Definition 4 is met automatically). The latter space can also be called (functorial) Berezin supermanifold or extended Berezin supermanifold. (Of course, for A we can also substitute Λ_n.)

The concept of Einstein-Grassmann algebra from the end of Section 7 is automatically transferred to the current situation. We will call the sheaf $\bar{\mathcal{A}}$ the Einstein-Grassmann sheaf if, for every $\bar{U}_i$, $\bar{\mathcal{A}}_{\bar{U}_i} \cong C^\infty(\bar{U}_i) \otimes \Lambda_n$, the algebra $C^\infty(\bar{U}_i)$ is an Einstein algebra. A supermanifold with an Einstein-Grassmann algebra as its differential structure will be called Einstein-Grassmann supermanifold.

In (Heller et al. 2024), we investigated a number of properties of functorial supermanifolds. In particular, curves in supermanifolds (super curves) exhibit rich properties which curves in an ordinary manifold are lacking. This may have important implications for the theory of singularities in superspace.

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References

Berezin, F. A. 1983. *Introduction to the Algebra and Analysis with Anticommuting Variables.* In Russian. Moscow University Press.

Berezin, F. A., and D. A. Leites. 1975. Supermanifolds. *Dokl. Akad. Nauk SSSR* 224(3):505–508.

DeWitt, B. 1984. *Supermanifolds.* Cambridge, London, New York: Cambridge University Press.

Geroch, R. 1972. Einstein algebras. *Commun. Math. Phys.* 26:271–275.

Heller, M. 1992. Einstein algebras and general relativity. *Int. J. Theor. Phys.* 31:277–288.

Heller, M., T. Miller, L. Pysiak, and W. Sasin. 2024. *Einstein-Grassmann superalgebras and the malicious singularity.* In preparation.

Heller, M., and W. Sasin. 1995. Sheaves of Einstein algebras. *Int. J. Theor. Phys.* 34:387–398.

Kock, A. 2006. *Synthetic Differential Geometry.* Cambridge: Cambridge University Press.

———. 2009. *Synthetic Geometry of Manifolds.* Cambridge: Cambridge University Press.

Kostant, B. 1977. Graded manifolds, gradet Lie theory, and prequantization. In *Lecture notes in math.* Vol. 570. Springer.

Moerdijk, I., and G. E. Reyes. 2010. *Models for Smooth Infinitesimal Analysis.* New York, Berlin, Heidelberg: Springer.

Navarro Goncàlez, J. A., and J. B. Sancho de Salas. 2003. *C^∞-Differentiable Spaces.* Berlin, Heidelber: Springer.

Nestruev, J. 2002. *Smooth Manifolds and Observables.* New York: Springer.

Palais, R. S. 1981. *Real Algebraic Differential Topology, Part I.* Publish or Perish.

Pysiak, L., W. Sasin, M. Heller, and T. Miller. 2020. Functorial differential spaces and the infinitesimal structure of space-time. *Reports Math. Phys.* 85(3):443–454.

———. 2023. Einstein algebras in a categorical context. *Reports Math. Phys.* 92(1):59–74.

Rogers, A. 2007. *Supermanifolds. Theory and Applications.* London, Singapore: World Scientific.