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ESTIMATION OF RISK NEUTRAL MEASURE FOR POLISH STOCK MARKET

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Abstract In the paper we present the application of risk neutral measure estimation in the analysis of the index WIG20 from Polish stock market. The risk neutral measure is calculated from the process of the options on that index. We assume that risk neutral measure is the mixture of lognormal distributions. The parameters of the distributions are estimated by minimizing the sum of squares of pricing errors. Obtained results are then compared with the model based on a single lognormal distribution. As an example we consider changes in risk neutral distribution at the beginning of March 2014, after the outbreak of political crisis in the Crimea.

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Introduction

One of the main element of contemporary financial theory is the risk neutral pricing. According to this rule the fair price of any financial instrument should be equal to the expected present value of its cash flow. However the expected value is calculated with regard to risk-neutral measure instead of the real probabilities. The risk neutral distribution (RND, also called martingale distribution) is the distribution representing expectation of the risk-neutral investor, i.e. the investor whose risk aversion is zero. On the market we have thus two separate probability measures: real probabilities P of possible future outcomes, and risk neutral probabilities Q, which contain information about market pricing. The relative density of those probability measures is called market price of the risk and reflects the risk aversion of the market as a whole. The prediction of this risk-free density has crucial implications from many perspectives, as it is forward-looking and reveals information about market expectations. The application of such predictions includes policy analysis, risk management and event studies.

As it was indicated by Breeden and Litzenberg (1978) the option prices contain information about risk-neutral distribution. Theoretically, if we knew prices of infinitely number of options with the same expiration date and with different strike prices, then we could calculate the density of risk-neutral distribution for this specific date. However in practice such a situation does not occur and some other methods of estimating

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risk-free measure should be used. The comprehensive survey of these methods can be found for example in (Bahra, 1997).

One application of RND is the search for arbitrage opportunities. As was indicated by Breeden and Litzenberg (1978), if such a density does not exist (for example probabilities are negative for some future prices) then there exists an arbitrage opportunity on the option market. With the butterfly spread strategy[1] (a portfolio of three options with different strike prices) one can replicate Arrow's contingent-state claim and obtain a gain without bearing any risk. The second application of RND is the analysis of market prediction. Bahra (1997) also indicated that the RND could have some value for monetary authorities. It was claimed that it can serve to estimate the efficiency of monetary policy and inflation expectations. Melick and Thomas (1997) have applied RND (estimated from American options) to crude oil market during Persian Gulf crisis. They found that the estimated distributions were consistent with the market commentary at those times and reflected a significant probability of severe disruption in the market for crude oil. The estimation of the RND was also applied in the analysis of the oil market in the works of Sadorsky (2001) and Gagnon and Power (2013).

Jackwerth and Rubinstein (1996) estimated risk-neutral density for the S&P 500 index based on European options. They found that after stock market crisis in the October 1987 the risk-neutral probabilities of huge decrease of the S&P 500 index had grown significantly and were bigger that the historical probabilities (estimated from time-series), which proves that the market expectation had changed. Mandler (2002) compared RND in the weeks in which meeting of the Governing Council of the European Central Bank took place to the weeks in which no meeting was scheduled. However the characteristics of RNDs in those two samples were highly variable and no systematic differences were found. Birru and Figlewski (2012) used RND in the analysis of financial crisis in the year 2008. Chabi-Yo, Garcia and Renault (2008) applied risk neutral probabilities to explain the equity risk premium puzzle[2] .Similarly Ziegler (2007) proposed the solution to volatility smile phenomenon, indicating that it can be an effect of aggregation of individual investors' RNDs in the situation where expectations of the investors differ. The RND can be also applied to the detection of jumps in the stock prices, as it was proposed by Wang (2009) and by Aït-Sahalia and Jacod (2009).

In the article we try to estimate RND for WIG20 index in the Polish stock market. The usage of the standard method of estimation poses some problems as the options in this market are illiquid. We thus use parametric approach with the possibly small number of parameters. We compare obtained RNDs with the densities in the Black-Scholes (lognormal) model and analyze the changes in the risk-aversion in the response to some market events.

Methods of recovering risk-neutral density from options' prices

Cox and Ross (1976) have shown that the price of a European option on the share S can be written as the discounted expected value of its payoffs at expiration. Thus the price of call option equals:

$$c(K,T)=e^{-rT}E^{Q}\left[\left(S_T-K\right)^{+}\right] \qquad (1)$$

where T is the expiration time, K is the strike-price of an option and r is the risk-free interest rate. The expectation is calculated with respect to some risk-neutral probability measure. If this measure has a density q_T (RND – risk neutral density) the price can be written as follows:

$$p(K,T)=e^{-rT}\int_0^K\left(K-x\right)q_T(x)dx \qquad (2)$$

The value of put option is

$$p(K,T)=e^{-rT}\int_0^K\left(K-x\right)q_T(x)dx \qquad (3)$$

Taking derivatives of both sides of (2) one obtains that

$$\frac{\partial c(K,T)}{\partial K}=e^{-rT}\left(1-Q_T(K)\right) \qquad (4)$$

where Q_T is cumulative distribution function of risk-neutral measure. According to equation (4) this measure can be calculated from call option prices with different strike prices. Taking again derivatives of both side of equation (4) we obtain that

$$q_T(x)=e^{rT}\frac{\partial^2 c(K,T)}{\partial K^2} \qquad (5)$$

1 See for example: Hull, J. (2009) Options, Futures and Other Derivatives, Prentice Hall, s. 225.

2 The empirical puzzle, why observed equity risk premia are hugely higher than the values predicted in the theory of finance. According to the observed risk premia the investors should be exceptionally risk-averse, what is inconsistent with other estimations. See: Mehra and Prescott (1985).

Similar dependences can be obtained for the prices of European put options. The RND can be computed from option prices. However in practice this is impossible, because in the real market there is only limited number of strike prices for any execution date. One can use discrete differences (instead of derivatives) or utilize the fact that butterfly spreads replicate contingent-state claims. Using the latter method the RND for future share price $S_T = K$ can be calculated with the following formula[3]:

3 See: Bahra (1997).

$$q_T(K) = e^{rT} \frac{c(K+\Delta K,T) + c(K-\Delta K,T) - 2c(K,T)}{\Delta K} \qquad (6)$$

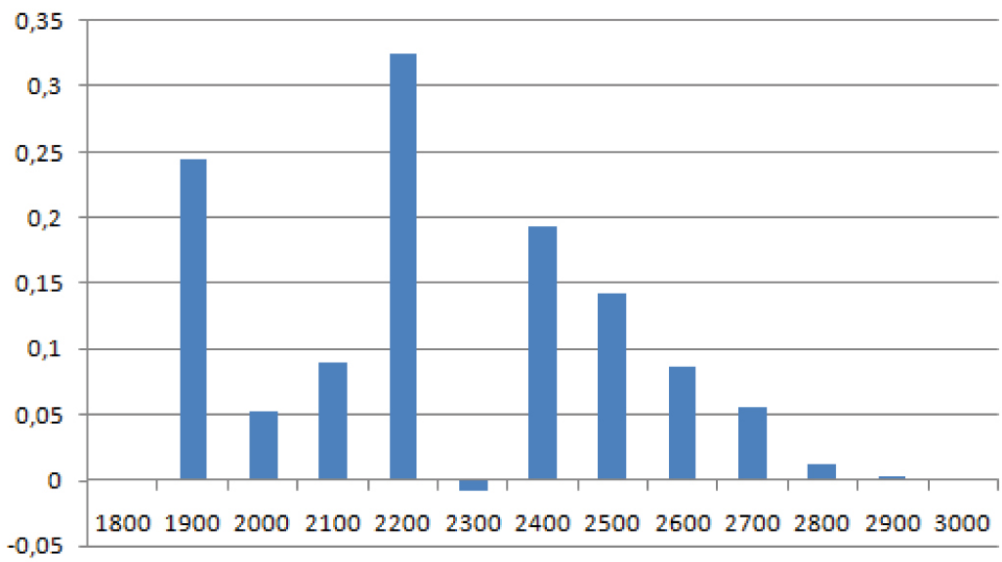
where $K-\Delta K$ and $K+\Delta K$ are strike prices next (higher and lower) to the strike price . However in practice this method is very unsatisfactory. The Figure 1 depicts the risk-neutral probabilities estimated with this method from the prices (quoted on 21 January 2014) of the options on WIG20 index with execution day 21 March 2014. For the price 2300 the probability is negative, which means that the options were mispriced and there existed an arbitrage opportunity. It was however too small to have any use in practice. The distribution has no proper shape (there are two peaks at the price 1900 and at the price 2200) and it is difficult to extrapolate the densities for the prices outside the range of strike prices.

In practice the RND is estimated from the option prices by specifying assumptions about density function and using some parametric or non-parametric methods. Aparicio and Hodges (1998) as well as Fusai and Roncoroni (2008) give a comprehensive survey of these methods.

Some methods consist on calibrating implied volatilities to the theoretical volatilities in the model. Rubinstein (1994), Jackwerth and Rubinstein (1996) and Jackwerth (1999) used implied binomial threes. Durpie (1994), Derma and Kani (1994) proposed a method base on utilizing volatility smile. Aït-Sahalia and Lo (1998) used kernel regressions. Another approach assumes that the price process of underlying instrument can be described by jump-diffusion models. This method was used by Hull and White (1987), Heston (1993), Bates (1991) and Wang (2009).

The method, commonly used in the practice of finance (in pricing derivative instruments and in risk analysis), consists on interpolating the volatility. The implied volatilities can be expressed as a function delta (Greek coefficient) and one searches for a function interpolating observed values. One can then obtain theoretical option prices using Black-Scholes formula. The method was used by Shimko (1993).

Figure 1: Risk-neutral probabilities for WIG20 index for expiration date March 21, 2014, calculated in January 21, 2014 from call option prices



Source: Author's own calculation

In this paper we use a parametric method based on lognormal distribution. We assume, as Melick and Thomas (1997) and Söderlind and Svensson (1997), that RND is some specific parametric function and then we try to find parameters of this function minimizing the differences between option prices in the market and theoretical prices. Melick and Thomas (1997) in their analysis of crude oil market during a crisis in Persian Gulf assumed that the RND is a mixture of three lognormal densities. In their approach those three distributions represented three possible outcomes of political crisis: i) a return to pre-crisis conditions, with Iraq peacefully withdrawing from Kuwait; ii) a disruption to Persian Gulf oil supplies, like damage to Saudi Arabian facilities during a war; iii) a continuation of unsettled conditions over the relevant horizon. Taking the mixture of n lognormal distributions requires estimating parameters and options in Polish market are illiquid, as there are days during which only a few options are traded. As we do not have enough data to estimate parameters of the mixture of three distributions, we take a mixture of only two lognormal distributions as a RND. The second reason for limiting the number of distributions in the mixture is the following. We try to analyze the reaction of the markets to Crimean crisis at the beginning of the March 2014. We see two possible outcomes of the political situation: either the situation could have worsened in the way that would have been harmful for the financial markets in the countries in the area (thus also in Poland) or the crisis could have ended in political détente. According to our assumption investors expected one of the two different outcomes, what gives rise to a mixture of two lognormal distributions.

The RND is given by the following function

$$q_T(x) = wf(x,\mu_1,\sigma_1) + (1-w)f(x,\mu_2,\sigma_2) \tag{7}$$

where

$$f(x,\mu_i,\sigma_i) = \frac{1}{x\sigma_i\sqrt{2\pi T}}\exp\left(-\frac{1}{2\sigma_i^2 T}\left(\ln x - \ln S_0 - \mu_i T + \frac{1}{2}\sigma_i^2 T\right)^2\right) \tag{8}$$

With such a risk-neutral measure the price of call option can be written as

$$\begin{aligned} c(K,T) = {} & e^{-rT} w\left(S_0 e^{\left(\mu_1+\frac{1}{2}\sigma_1^2\right)T} N(d_{11}) - KN(d_{21})\right) + \\ & + e^{-rT}(1-w)\left(S_0 e^{\left(\mu_2+\frac{1}{2}\sigma_2^2\right)T} N(d_{12}) - KN(d_{22})\right) \end{aligned} \tag{9}$$

and the price of put option is

$$\begin{aligned} p(K,T) = {} & e^{-rT} w\left(KN(-d_{21}) - S_0 e^{\left(\mu_1+\frac{1}{2}\sigma_1^2\right)T} N(-d_{11})\right) + \\ & + e^{-rT}(1-w)\left(KN(-d_{22}) - S_0 e^{\left(\mu_2+\frac{1}{2}\sigma_2^2\right)T} N(-d_{12})\right), \end{aligned} \tag{10}$$

where

$$d_{1i} = \frac{\ln S_0 - \ln K + \left(\mu_i + \frac{\sigma_i^2}{2}\right)T}{\sigma_i\sqrt{T}}, \quad d_{1i} = d_{2i} - \sigma_i\sqrt{T} \tag{11}$$

The parameters in the models are calculated by minimizing the sum of the squared differences between option prices in the market and the theoretical prices, given by the equations (9) and (10). Assume that at the specific date n call options with strike prices K_1, K_2, …, K_n were traded and the prices of those options were c_1, c_2, …, c_n. On the same day there was a trade for m put options with strike prices H_1, H_2, …, H_m and the quoted prices of options were equal to p_1, p_2, …, p_m. All call and put options had the same expiration time T. We obtain the parameters $\theta = (\mu_1, \mu_2, \sigma_1, \sigma_2, w)$ of the RND for the moment by solving the following problem:

$$\min_{\theta}\left[\sum_{i=1}^{n}\left(c_i - c(K_i,T)\right)^2 + \sum_{i=1}^{m}\left(p_i - p(H_i,T)\right)^2\right] \qquad (12)$$

There is no analytical solution to this problem, but it can be solved numerically using standard algorithms for non-linear optimization.

Risk-neutral measure for the WIG20 index

We have estimated the risk-neutral measure based on the two series of options on WIG20 index[4]. The first series expired on 21st of March 2014 and the second series has an expiration date 20th of June 2014. Thus the RND (and market expectations concerning future outcomes) were calculated for those two days. In both series strike prices for call and put options varied from 1800 to 3000. The values of WIG20 index in the period from July 2013 to the end of March 2014 varied in the range from 2200 to 2600. The market for options was not very liquid. For both series there were trading days when there was no trade of any option in the series. We have calculated martingale measures only for days in which at least 10 options in the series were traded.

For each of those days we have solved the problem (12) and obtained parameters of RND. The computations were performed in the R statistical package[5] and optimization was done with the Nelder-Mead algorithm[6]. To compare our approach with simpler model we have also calculated the parameters of single lognormal distribution as a RND. This allows us to compare model based on mixture of lognormal distributions with a standard Black-Scholes model. The parameters of lognormal model were also calculated by minimizing sum of squares of differences between the theoretical options' prices and the observed ones. It turned out that model based on mixture of distributions has better fit for the both series of options. For the March series the mean value of minimized function (sum of squares of errors) in lognormal model was 1700 and in the model based on mixture this value was 826. For the June option series these numbers were, respectively, 8448 and 4541. The mean square error (MSE) in the lognormal model was 9.98 for March series and 24.51 for June series. The MSE in the mixture model was 7.07 (March series) and 18.48 (June series). The models are nested but the parameters were not estimated with maximum likelihood method, so formally we cannot use standard likelihood ratio test for nested models to decide whether the more general model (based on the mixture) statistically better fits to the data. We can only mention that the test statistic, assuming normal distribution of pricing errors was 256 for March series and 205 for June series, while the critical value at the significance level 0.001 is 10.8, which would mean that one should abandon null hypothesis that there is no statistical difference between models and decide that the more general model is better. The difference is high and we are convinced that the change in the estimation method (from minimizing sum of squares to maximizing likelihood) would not change these figures so much that the likelihood ratio test would give different results.

It is worth mentioning that the risk neutral distributions calculated in the model with mixture of measures and in the model of lognormal distribution share very similar characteristics. For example, the expected values and standard deviations of future values of WIG20 were very similar in both models. The third and four moments (skewness and kurtosis) in both models were also very close to each other. The main advantage of the model with the mixture of the distributions lies in the fact that it allows for

4 The index of twenty largest companies on the Polish stock market. As for now (year 2014) it is the sole underlying instrument for options traded on the Polish Stock Exchange.
5 See (R Core Team, 2013).
6 See (Nelder & Mead, 1965).

two different future outcomes. The difference in the performance of the models suggests that this is exactly the way in which the financial market perceives the future.

Tables 1 and 2 contain results of the estimations for the model in which RND is the mixture of two lognormal distributions. In the Table 1 there are mean values of the parameters of RND (a mixture of two lognormal distributions) for 21st of March 2014 for each month within the sample[7]. Table 2 contains the parameters of RND for 20th of June 2014. To investigate the changes in the perceptions of the markets' view of the future, we have split the samples into several sets, each of them containing observations for a single month. The tables 1 and 2 contain the means of the estimated parameters for every month in the sample.

7 Note that the RND is the distribution of WIG20 index for the day of exercise (here on 21st of March). However the distribution is calculated in specific day prior to exercise. It reveals the market expectations in this day about index level in the exercise day. Thus the first row in the Table 1 contains information about expectations in July concerning prices in March 2014. The second row reveals what market expected in August to happened with prices in March, etc.

Table 1: Parameters of the RND for the March'14 option series

Month	*μ1*	*μ2*	*σ1*	*σ2*	*W*
July 2013	-0.147	0.070	0.216	0.125	0.495
August 2013	-0.121	0.057	0.213	0.114	0.480
September 2013	-0.264	0.120	0.226	0.130	0.414
October 2013	-0.306	0.120	0.176	0.116	0.355
November 2013	-0.457	0.159	0.118	0.111	0.321
December 2013	-0.598	0.193	0.090	0.127	0.287
January 2014	-0.662	0.293	0.106	0.128	0.340
February 2014	-1.351	0.271	0.117	0.134	0.246
March 2014	-6.244	0.997	0.210	0.100	0.350

Source: Author's own calculations

Table 2: Parameters of the RND for the June'14 option series

Month	*μ1*	*μ2*	*σ1*	*σ2*	*w*
December 2013	-0.769	0.574	0.226	0.184	0.533
January 2014	-0.818	0.658	0.308	0.217	0.546
February 2014	-1.317	0.888	0.503	0.223	0.514
March 2014	-5.696	1.361	1.419	0.775	0.361

Source: Author's own calculations

In our interpretation the two lognormal distributions in the mix represent investors' view about two possible outcomes in the economy. As one can see in the Tables 1 and 2 expected values in the first distribution ( ) were lower than in the second distribution of the mixture ( ). On the other hand the standard deviations of the first element of the mixture is usually higher ( ). We can thus pose an interpretation of these two distributions in the mixture. The first component represents the "bad" state of the economy, with lower returns and higher volatility, whereas the second component represents the state of the nature with developing economy and good situations in the financial market. The value is the probability, as perceived by the market, that the situation will end in the "bad" state before the date of exercise.

One can extract much of the information contained in the RND by calculating some summary statistics. For example the mean gives the information about the expectations of the future values of the underlying instrument. The same information can be also captured from median of mode of the distribution. The standard deviation gives information about the uncertainty concerning future prices. The third moment characterizes the distribution on both sides of the mean. It gives information, if the market expects rather higher downfalls (negative skewness) or higher upward jumps of the prices (positive skewness). The fourth moment represents the market expectations of extreme events. If the kurtosis is bigger, the market expects that there will be more extreme changes in the prices. The tables 3 and 4 give the main characteristic of the RND for future value of WIG20 index. In the both tables we present the expected value, standard deviation, skewness and excess kurtosis (i.e. the kurtosis minus 3, characteristic for normal distribution). All those characteristics are presented for the risk neutral distribution of WIG20 values, so the value 2261.9 in the first row of the Table 3 is the July's market expectation of the value of WIG20 index in March 2014. Similarly the figure 380.2 in the first row denotes the uncertainty of these expectations.

Table 3: Characteristics of the RND for the March'14 option series

Month	*Expected value*	*Std.*	*Skewness*	*Excess kurtosis*
July 2013	2261.9	380.2	60.5	0.5
August 2013	2390.8	352.3	80.3	1.0
September 2013	2338.9	366.4	-1.0	0.4
October 2013	2477.0	311.1	-32.0	0.2
November 2013	2555.7	281.5	-81.4	-0.3
December 2013	2446.2	256.1	-72.4	-0.4
January 2014	2369.2	209.0	-46.6	-0.6
February 2014	2455.1	163.5	-95.4	0.5
March 2014	2395.1	119.6	-172.1	4.7

Source: Author's own calculations

Table 4: Characteristics of the RND for the June'14 option series

Month	*Expected value*	*Std.*	*Skewness*	*Excess kurtosis*
December 2013	2403.2	566.0	133.4	-0.4
January 2014	2349.0	440.1	57.9	-0.7
February 2014	2438.3	369.7	-9.6	0.8
March 2014	2377.8	324.4	-45.1	1.1

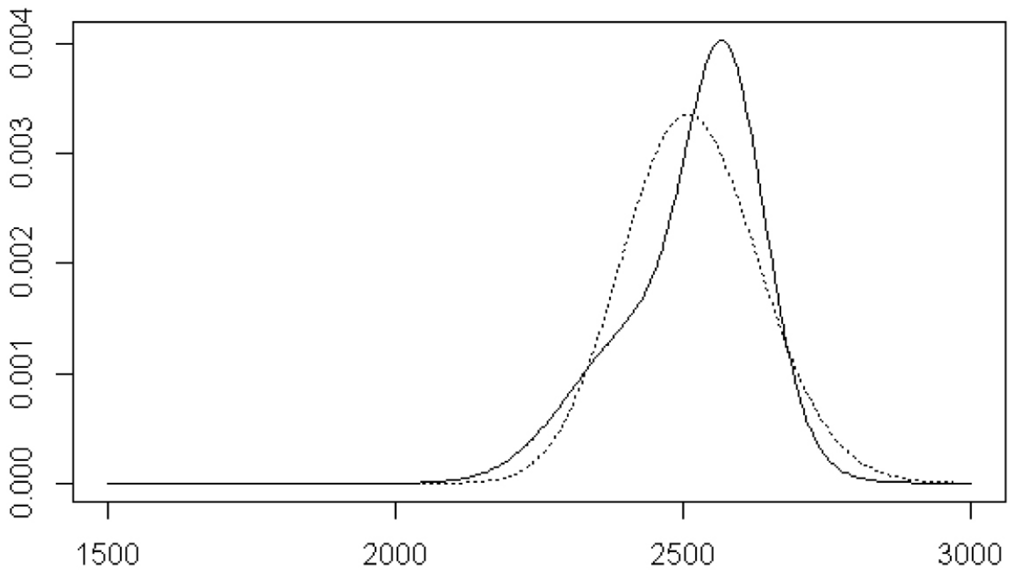
Source: Author's own calculations

One of the main observations from the tables 3 and 4 is the fact that starting from November there was a steady decline in the expected value of WIG20 (both for March and for June). The other significant feature is the decline of skewness and rise of kurtosis at the beginning of 2014.

As one can see in the tables 3 and 4 in the March 2014 the expected value of RND falls and in this month there was an unprecedented fall of skewness and noticeable rise in the kurtosis – for both series of options. This result can be verified with the tables 1 and 2. The difference , which denote in the expected rates of returns in two possible states of the world, grew significantly in the March 2014. Notice that there was also an increase in the differences of standard deviations in the distributions concerning two possible outcomes; the value increased from -0.017 in February to 0.110 in March (March option series). For the June's series the increase was from 0.280 in February to 0.664 in March. This divergence was connected with the worsening position of the "bad" possible state of the world, i.e. the first state. The mean return for this state has fallen and the standard deviation has risen, while in the "good" state of the economy the parameters have not changed so much.

This brings us to one of the main application of RND estimates, namely to the detection of the unusual events on the markets. The beginning of the year 2014 was a period of rising political conflict between the Russian Federation and the Ukraine. The beginning of March is the outburst of the crisis in Crimea. We are fairly convinced that the changes in the characteristics of the RND prices are strictly connected with this event. We have screened the shapes of distribution of RND for the trading days at the beginning of the year 2014 and have found that there were some major changes in those days in which the political tension in the region grew. Especially interesting is the period at the very beginning of the March. On the 1st of March the Federation Council of Russia approved the use of the Russian troops in the Crimea. It was on Saturday when markets were closed. The Figure 2 depicts the RND in the last trading day before this event, namely on 28th of February. The solid line is the density of risk-neutral measure in the mixture model, while the dotted line is the density of lognormal risk-free density. The Figure 3 contains the estimated risk-neutral densities in the next trading day, namely 3rd of March 2014 (Monday). Again, we present the densities in the lognormal model (dotted line) and in the model based on the mixture of lognormal distributions (solid line). Comparing these two days and the two models we can draw two conclusions. Firstly, there was a huge difference in the risk-neutral distribution in those two subsequent trading days. The RND shifted significantly to the left and it can be seen in the model with one lognormal distribution as well as in the model with the mixture of distributions. The second observation is that the change is more visible in the mixture model. In the lognormal model the peak of the distribution moved only slightly to the left (although there was a change in the dispersion of the distribution), whereas the distribution based on the mixture of the lognormal distributions shifted dramatically to the left (the peak moved from a little above 2400 to the value a little above 1600) and the dispersion significantly diminished. This suggests that the model based on the mixture of distributions can serve better in the detection of significant changes in the markets' expectations concerning future.

Figure 2: Risk neutral density on 28th of February 2014
(solid line – model with mixture of distributions, dotted line – lognormal distribution)



Source: Author's own computations

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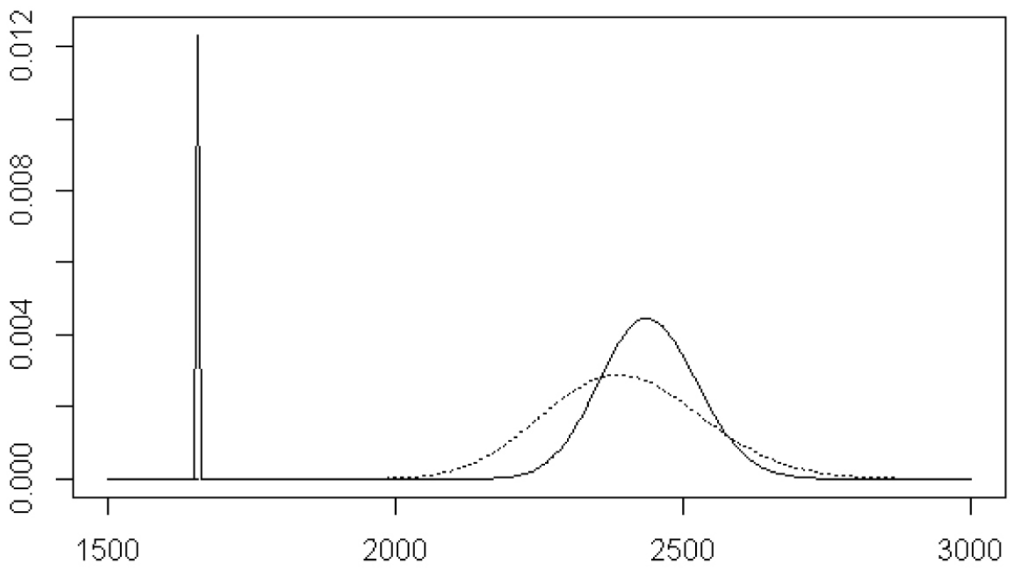
Figure 3: Risk neutral density on 3rd of March 2014
(solid line – model with mixture of distributions, dotted line – lognormal distribution)



Source: Author's own computations

Conclusions

In the article we use European option prices to estimate the market's probability distribution for the index WIG20 from the Polish stock exchange. As the form of risk-neutral distribution we use the mixture of two lognormal distributions. This particular assumption was driven by the conditions in the market at the beginning of the year 2014 during the outburst of the Crimean crisis. The second reason for assuming such a model was the fact that the market for options was no liquid and thus we were forced to choose a model with possibly few parameters. We have shown that the methodology presented here should be useful for those, who wish to utilize data from options' market and try to search for the structural changes in the market's expectation concerning the future prices. We have indicated that the model of risk-neutral measure based on the mixture of lognormal distributions gives a better performance in this respect than a model based on a single lognormal distribution. The mixture model with two distributions has sufficiently few parameters to be estimated from the observations based on several options.

In the application to the analysis of the Polish stock market at the turn of years 2013 and 2014, we have found that there was significant change in the expectations in the market. Since November 2013 the expected values of the WIG20 in March and June have lowered significantly. There was also a drop of the skewness and the rise of the kurtosis, which indicated that markets were beginning to expect large price changes (kurtosis) and the direction of these changes was expected to be negative (negative skewness).

Finally, examination of particular days confirmed that the changes in the risk-neutral densities are connected with some extreme events. Particularly important was the case of the changes between 28th of February and 3nd of March 2014 where the markets can react only on Monday to the political event that took place on Saturday.

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