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LOG-PERIODIC POWER LAW AND GENERALIZED HURST EXPONENT ANALYSIS IN ESTIMATING AN ASSET BUBBLE BURSTING TIME

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Abstract

We closely examine and compare two promising techniques helpful in estimating the moment an asset bubble bursts. Namely, the Log-Periodic Power Law model and Generalized Hurst Exponent approaches are considered. Sequential LPPL fitting to empirical financial time series exhibiting evident bubble behavior is presented. Estimating the critical crash-time works satisfactorily well also in the case of GHE, when substantial „decorrelation" prior to the event is visible. An extensive simulation study carried out on empirical data: stock indices and commodities, confirms very good performance of the two approaches.

JEL classification: C22, C53, C61
Keywords: asset bubble, crash, Log-Periodic Power Law, Generalized Hurst Exponent, multifractality, forecasting, bursting time estimation

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Introduction

Speculative bubbles have been occurring all throughout the history of financial markets, irrespective of the asset classes involved. One of the most pronounced pioneer bubbles was the Dutch tulip mania in 1635-1637, followed by a huge crash that wiped out large fortunes. Whereas the early stages of bubble formation usually pass unnoticed, the ripe phases of these anomalies can be detected by a number of techniques, eg. augmented Dickey-Fuller tests for unit root. In addition, taking present market fundamentals simultaneously into account usually makes the work more successful. Bubbles inevitably burst, leading to severe price downturns or outright crashes of magnitude corresponding to the scale of the preceding overvaluation. The exact moment of this bursting, called also a crash-time or rupture point, draws a clear line between two distinct regimes for price dynamics. As far as investment efficiency is concerned, predicting the crash-time t_c poses a financially vital and mathematically challenging research problem. Its importance is associated with large financial bets put at stake, especially shortly before the crucial peak.

Several approaches have been proposed to model the price dynamics prior to and right after the bust. One of the powerful tools has been developed and expanded for nearly two decades by D. Sornette, who employs a Log-Periodic Power Law for modeling the asset price dynamics (Johansen, Ledoit & Sornette, 2000). Another precursor of this approach is S. Drożdż (Drożdż, Grummer, Ruf & Speth, 2003). Importantly, although the very time tc can be easily determined ex-post, one should rather focus upon its reliable interval estimation as the whole process of bubble bursting can be interpreted as a phase transition. The method has been proved successful on a number of occasions (Zhang et al., 2016), e.g. spectacularly precise prediction of crude oil bubble bursting time in 2008 (Drożdż, Kwapień & Oświęcimka, 2008).

Another promising tool for detecting the end of the speculative bubble is analysis of long range dependence using the Hurst exponent. Specific decorrelation envisaged in investor behavior can be found in numerous papers such as those by: Kristoufek (2010), Grech and Pamuła (2008), Morales, Di Matteo, Gramatica, Aste (2012).

Although there exist other concurrent tools devised to tackle this topic (dynamic systems evolution, smooth transition models), in this paper we focus on the two above, applicationally vital approaches, useful both from an academic and business point of view.

Log-Periodic Power Law Model

In our first approach to detect speculative bubbles we are using the LPPL model. Based on Johansen et al. (2000) we assume that in a bubble regime price follows a stochastic differential equation:

$$\frac{dp}{p} = \mu(t)dt + \sigma(t)dW - \kappa dj \quad (1)$$

where $p = p(t)$ is the asset stock price, $\mu(t)$ - drift, $\sigma(t)$ - volatility, dW is the increment of a standard Wiener process and dj represents a discontinuous jump such that $j = 0$ before the crash and $j = 1$ after the crash. Each successive crash corresponds to a unit jump of j. The parameter κ quantifies the amplitude of the crash when it occurs.

Denote F_t - filtration generated by the price process $p(t)$, namely $F_t = \sigma\{p(s): s \leq t\}$. The jump's dynamics are governed by a crash hazard rate $h(t)$. Since $h(t)dt$ is the probability that the crash occurs between t and $t + dt$ conditionally on the fact that it has not yet happened, we have

$$E(dj \mid F_t) = 1 \times h(t)dt + 0 \times (1 - h(t))dt = h(t)dt \quad (2)$$

The JLS model assumes that two types of agents are present on the market: a group of traders with rational expectations and a group of noise traders who exhibit herding behavior that may destabilize the asset price. According to this model, the actions of noise traders are quantified by the following dynamics of the hazard rate (Johansen et al., 2000):

$$h(t) = B'(t_c - t)^{m-1} + C'(t_c - t)^{m-1}\cos(\omega \ln(t_c - t) - \varphi') \quad (3)$$

where B', C' denote amplitude parameters; ω, φ' - phase parameters and t_c is the critical time marking the end of the bubble. The power law behavior $(t_c - t)^{m-1}$ embodies the mechanisms of positive feedback at the origin of the bubble formation. The log-periodic function $\cos(\omega \ln(t_c - t) - \varphi')$ takes into account the existence of a possible hierarchical cascade of panic acceleration causing the bubble to pop.

In the JLS model the rational agent is risk neutral and has rational expectations. Thus, the asset price $p(t)$ follows a martingale process: $E(p(t')|F_t) = p(t) \ \forall t' > t$.

Under these assumptions the no-arbitrage condition is just $E(dp|F_t) = 0$.Accordingly, the excess return $\mu(t)$ is proportional to the crash hazard rate, namely $\mu(t) = \kappa h(t)$ Thus, solving (1) under the condition that no crash has occurred yet $(dj = 0)$ leads to the following log-periodic power law (LPPL) equation for the log-price expectation:

$$E[\ln p(t)] = A + B(t_c - t)^m + C(t_c - t)^m \cos(\omega \ln(t_c - t) - \varphi) \quad (4)$$

where $A = \ln p(t_c)$ $B = -\kappa B'/m$ and $C = -\kappa C' / \sqrt{m^2 + \omega^2}$. It should be noted that solution (4) describes the dynamics of the average log-price only up to the critical time t_c and cannot be used beyond it. This crash-time t_c corresponds to the termination of the bubble and indicates the change to another regime, which could be either a large crash with accelerating oscillations (negative bubble – Wątorek, Drożdż & Oświęcimka, 2016) or decelerating oscillations (anti-bubble – Johansen & Sornette, 1999) or a change of the average growth rate.

The LPPL model (4) is described by 3 linear parameters (A,B,C) and 4 nonlinear parameters (m,ω,t_c,φ). These parameters are subject to the following constraints, described in Sornette, Woodard, Jiang, Zhou (2013): $0 < m < 1; 2 \leq \omega \leq 22; B < 0, |C| < 1, t \leq t_c$.

To fit the LPPL function (4) to empirical data we employed a procedure proposed by Filimonov and Sornette (2013), which reduces the estimation to just three nonlinear parameters t_c,m,ω. The key idea of this method is to decrease the number of nonlinear parameters and simultaneously to eliminate the interdependence between the phase φ and the angular log-frequency ω. Let us rewrite (4) by expanding the cosine term as follows:

$$\ln E[p(t)] = A+B(t_c-t)^m + C(t_c-t)^m \cos(\omega \ln(t_c-t))\cos\varphi + C(t_c-t)^m \sin(\omega \ln(t_c-t))\sin\varphi \quad (5)$$

Now, we introduce two new parameters:

$$C_1 = C \cos\varphi, \qquad C_2 = C \sin\varphi \quad (6)$$

and rewrite the LPPL equation (4) as

$$\ln E[p(t)] = A+B(t_c-t)^m + C_1(t_c-t)^m \cos(\omega \ln(t_c-t)) + C_2(t_c - t)^m \sin(\omega \ln(t_c - t)) \quad (7)$$

As seen from (7), the LPPL function has now only 3 nonlinear (t_c,ω,m) and 4 linear (A,B,C_1,C_2) parameters, and the two new parameters C_1 and C_2 contain the former phase φ.

The resulting model is calibrated on the data using the Ordinary Least Squares method, providing estimators of all the parameters: t_c,m,ω,A,B,C_1,C_2 within a given time window subject to analysis.

GHE approach

In our second approach, we aim at connecting "decorrelation" (long memory tapering) and multifractality growth with the bursting of the speculative bubble. To achieve that, we employed the notion of Generalized Hurst Exponent, henceforward GHE, based on Di Matteo (2007). This exponent is a tool for studying directly the scaling properties of the data via the *q*-th order moments of the distribution of the time series increments $S(t)$ with $1 \leq t \leq T$, namely:

$$K_q(\tau) = \frac{1}{T - \tau + 1} \sum_{t=0}^{T-\tau} \left|S(t+\tau) - S(t)\right|^q \quad (8)$$

where $\tau \in \{\tau_1,...,\tau_{max}\}$. GHE is then obtained from the scaling behavior of function (8) when the following relation holds:

$$K_q(\tau) \sim \tau^{qH(q)} \quad (9)$$

and hence we calculate GHE via regression from the following function:

$$K_q(\tau) = C\tau^{qH(q)} \quad (10)$$

Processes exhibiting this scaling behavior can be divided into two classes:

1) Processes with $H(q) = H$, i.e. independent of q These processes are unifractal, which means that their scaling behavior is uniquely determined by the constant *H*, known as Hurst exponent or self-affine index (Di Matteo, 2007).

2) Processes with non-constant $H(q)$ are called multiscaling (or multifractal) and each moment scales with a different exponent. Previous works have pointed out how financial time series exhibit scaling behaviors which are not simply fractal, but rather multifractal, e.g. Di Matteo (2007).

Depending on *q*, the exponents are associated with special features. For instance, when $q = 1$, $H(1)$ describes the scaling behavior of the absolute values of the increments. This exponent value is expected to be closely related to the original Hurst exponent H indeed scaling the absolute spread within the increments. The exponent value corresponding to $q = 2$ is associated with the scaling of the autocorrelation function and is related to the power spectrum index (Di Matteo, 2007).

Quite intuitively, the recent past is more important than the remote past. To incorporate that we can assume that the informational impact of observations decays exponentially. This smoothing can be attained by defining weights as:

$$w_t = w_0 \exp(-t/\theta); \; 0 \le t \le T \tag{11}$$

where $\theta \ge 0$ is the weights characteristic time. Introducing an exponential decay factor $\alpha = 1/\theta$, the parameter w_0 is given by Pozzi, Di Matteo, Aste, (2012) as

$$w_0(\alpha) = \frac{1 - \exp(-\alpha)}{1 - \exp(-\alpha T)} \tag{12}$$

Hence the weighted GHE (abbrev.: GHEw) is obtained by replacing normal averages in (8) with weighted averages

$$K_q^w(\tau) = \frac{1}{T - \tau + 1} \sum_{t=0}^{T-\tau} \left| S(t+\tau) - S(t) \right|^q w_t \tag{13}$$

as described in the scaling relation (9), so we get

$$\ln K_q^w(\tau) = \ln C + qH^w(q) \ln \tau \tag{14}$$

As an indicator of degree of multifractality we consider the quantity:

$$\Delta H(1,2) = H(1) - H(2) \tag{15}$$

following the paper of Morales, Di Matteo, Aste (2014).

Multifractality in time series can be interpreted as a consequence of fat-tailed behavior. Study given in Barunik, Aste, Di Matteo, Liu (2012) shows that temporal correlations have the effect of reducing the measured multifractality. In Moreales et al. (2012) GHEw was used as a tool to detect unstable periods within financial time series.

In our analysis we want to link multifractality increase at the end of the speculative bubble both with autocorrelations decay and fat-tailed distributions. According to Fractal Market Hypothesis, multifractality in time series may result from the existence of multiple market players having different time horizons (Weron & Weron, 2000). Capturing the dynamics of investors' interactions can be carried out by using GHE, which measures the autocorrelation function decay rate.

Empirical results

LPPL approach

In order to fit the LPPL function described above we have to select the initial parameters t_c, m, ω. Next, we need to calculate linear parameters A, B, C_1, C_2 by OLS method and then minimize the cost function using nonlinear least squares method. In previous works random choice of the initial parameters was proposed, see e.g. Filimonov and Sornette (2013), using local peak detection Pele (2012) or constant ω – Drożdż et al. (2003). In our work we decided to test all possible values of startup parameters t_c, m, ω, $m \in [0.1, 0.95]$ with step 0.05, $\omega \in [2,22]$ with step 0.5, $t_c \in [t{+}1, t{+}101]$ or $t_c \in [t{+}1, t{+}151]$ with step 10, depending on the data length. All calculations were performed in Matlab package. We minimized the cost function by using *Region-Trust* algorithm, which in Bree and Joseph (2013) was proposed as an improvement to the traditionally used *Levenberg-Marquardt* algorithm.

To get more robust results, we carried out the analysis on empirical data with moving starting point with step either 5 or 10 trading days in a shrinking time window $[t_1, t_2]$ and moving end point with 5 trading days step in an expanding time window $[t_1, t_2]$, $t_2 < t_c$, similar as in Jiang, Zhou, Sornette, Woodard (2010). For each time window approximately 8000 combinations of the prescribed initial parameters were taken and after the nonlinear optimization we got the same amount of parameters with the sum of squared residuals. Lowest SSR points at the best t within each time window. During the fitting process getting a stable value of t_c is essential, therefore we compared SSR's from each time window by counting mean squared error. Finally, we calculated an 80% confidence interval based on 5% t_c with the lowest MSE.

We tested our approach on 10 historical asset bubbles and then applied it on current, real financial time series. The results are presented in the table below:

Table 1: LPPL fitting results

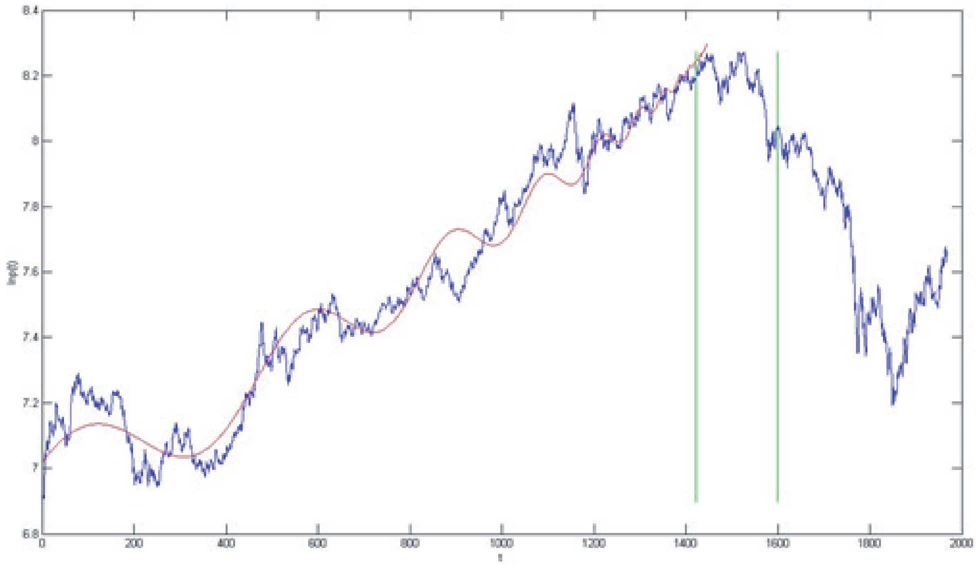
data	t_1	t_{end}	n	best t_c	t_c interval	peak
SHX	1/13/2014	10/2/2015	421	**347**	**319-371**	346
DAX	8/22/2011	10/2/2015	1046	**925**	**930-1004**	923
DJI	9/15/1981	8/21/1987	1502	**1477**	**1464-1524**	1504
DJI	7/23/2002	10/5/2007	1312	**1359**	**1260-1360**	1314
DJI	12/15/1920	9/4/1929	2259	**2245**	**2244-2336**	2259
WIG20	10/3/2001	10/29/2007	1526	**1446**	**1422-1598**	1526
Nasdaq	10/22/1998	3/13/2000	345	**376**	**366-426**	379
Nikkei	10/9/1981	12/29/1989	2043	**2043**	**2036-2129**	2043
Gold	2/2/2001	8/31/2011	2666	**2666**	**2659-2748**	2669
HSX	9/30/2002	10/30/2007	1258	**1240**	**1235-1269**	1258
Silver	10/14/2008	4/28/2011	649	**654**	**623-661**	649
CL	10/6/2006	7/3/2008	441	**472**	**421-543**	441

Source: Authors' own computations

Here t_1, t_{end}, n denote the time series beginning, end and length, respectively; t_c interval - 80% confidence interval based on 5% t_c's with the lowest MSE.

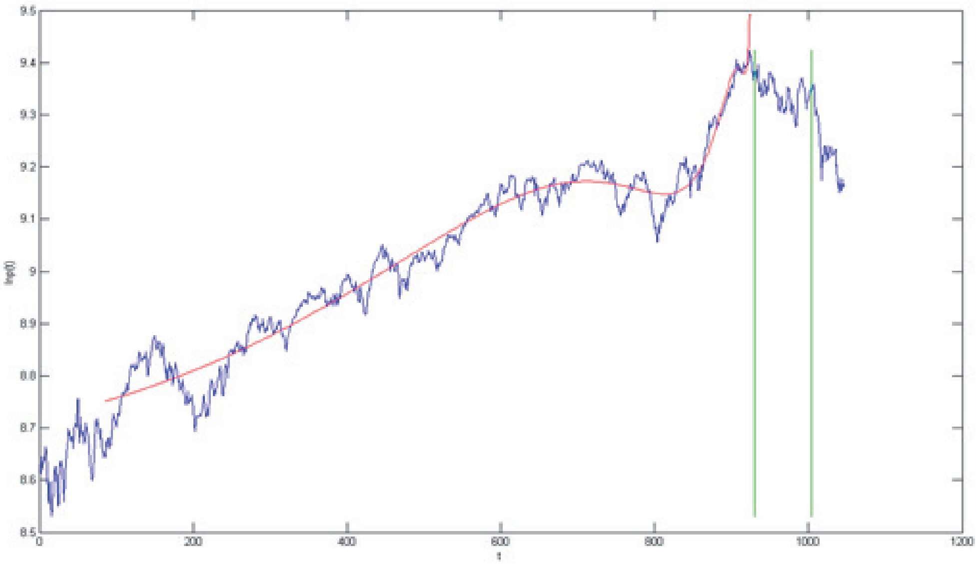
Figures 1–3 below show three stock indices from Table 1, namely WIG20, DAX and Shanghai Composite. Best LPPL function is depicted in red and the 80% t_c confidence interval in green.

Figure 1: WIG20 3.10.2001-29.10.2007; best fit: t_c=1446, m=0.8001, ω=14.0023



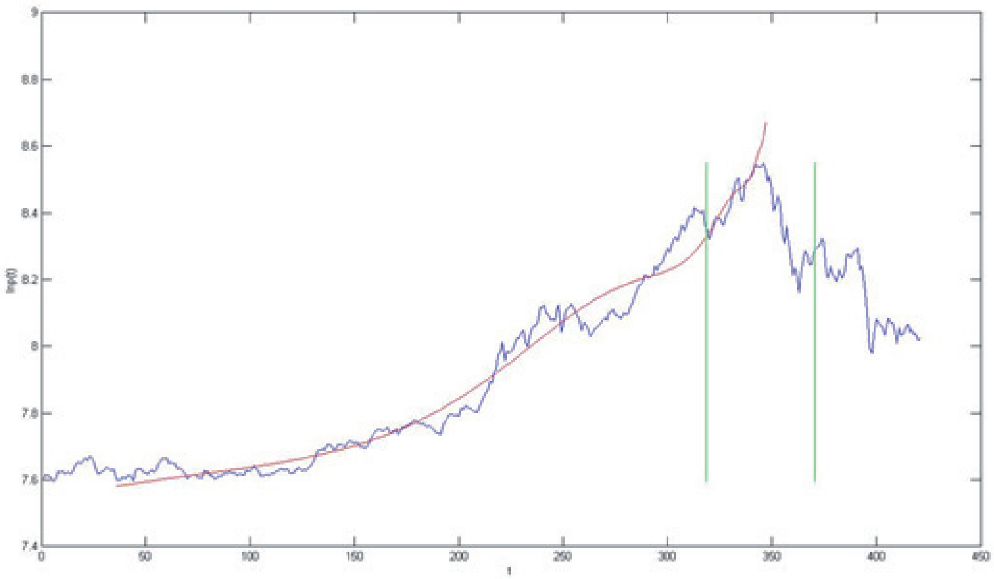
Source: Authors' own computations

Figure 2: DAX 22.08.2011-02.10.2015; best fit: t_c=925, *m*=0.2999, *ω*=2.4988



Source: Authors' own computations

Figure 3: SHX 13.01.2014-02.10.2015; best fit: t_c=347, *m*=0.5498, *ω*=5.9967



Source: Authors' own computations

It is clearly seen that in all historical cases fitting the LPPL function leads to good tc predictions. Results are comparable with those of Johansen and Sornette (2010). Especially worth noting, in April 2015 we achieved a perfect bubble peak detection for DAX, when the regime change occurred just 2 days prior to our estimated t_c. We also made post-hoc analysis of a recent bubble burst in the Shanghai Composite index, obtaining almost perfect prediction.

The above results convincingly show that the LPPL approach is a very reliable tool for ex-ante predicting the bubble peak moment t_c. This technique is undergoing further dynamic development. One of the possible generalizations could be higher-order expansion of LPPL functions, addressed also to high frequency data analysis (on an intraday basis crucial news can sometimes lead to major reversals).

GHE approach

In our second approach we calculated GHEw from equation (14), using the open source algorithm of T. Aste (Moreales et al., 2014) in the Matlab package. We set up $\tau \in \{5,\ldots,19\}$, according to Di Matteo (2007). The time windows used for successive GHEw calculations contain 250 data points (approximately T = 250 working days in a year), each window is shifted by one trading day, t = 1. According to Pozzi et al., (2012), we used exponential smoothing lag equal to θ = 83 days (around three months). Working with log-prices, the resulting GHEw estimates correspond to the log-returns.

We used our second approach on the same empirical data and we obtained the following results:

Table 2: GHEw results

data	*t1*	*tend*	*n*	min *H*(1)	*H*(1)	max ΔH	*H*(1)-*H*(2)	peak	crash
SHX	1/13/2014	10/2/2015	421	**389**	0,4943	**355**	0,083	346	351
DAX	8/22/2011	10/2/2015	1046	**1003**	0,4766	**948**	0,0523	923	1017
DJ	9/6/1983	12/31/1987	1093	**1012**	0,3818	**972**	0,0735	1004	1033
DJI	7/19/2004	3/31/2009	1185	**729**	0,2664	**894**	0,081	814	1059
DJI	12/16/1924	11/25/1929	1323	**1273**	0,4588	**1241**	0,0857	1259	1305
WIG20	10/3/2003	5/30/2008	1169	**1013**	0,3506	**1032**	0,0652	1026	1080
Nasdaq	9/10/1998	3/16/2001	635	**356**	0,3506	**354**	0,0357	379	395
Nikkei	1/8/1986	4/5/1990	1106	**1048**	0,3829	**1009**	0,041	1043	1079
Gold	2/2/2001	12/30/2011	1252	**976**	0,343	**1135**	0,0916	1169	none
HSX	10/6/2003	2/25/2009	1081	**843**	0,4091	**904**	0,0793	758	973
Silver	10/14/2008	11/11/2011	790	**576**	0,3432	**652**	0,0917	649	654
CL	10/6/2006	10/23/2008	519	**483**	0,3491	**432**	0,0426	441	501

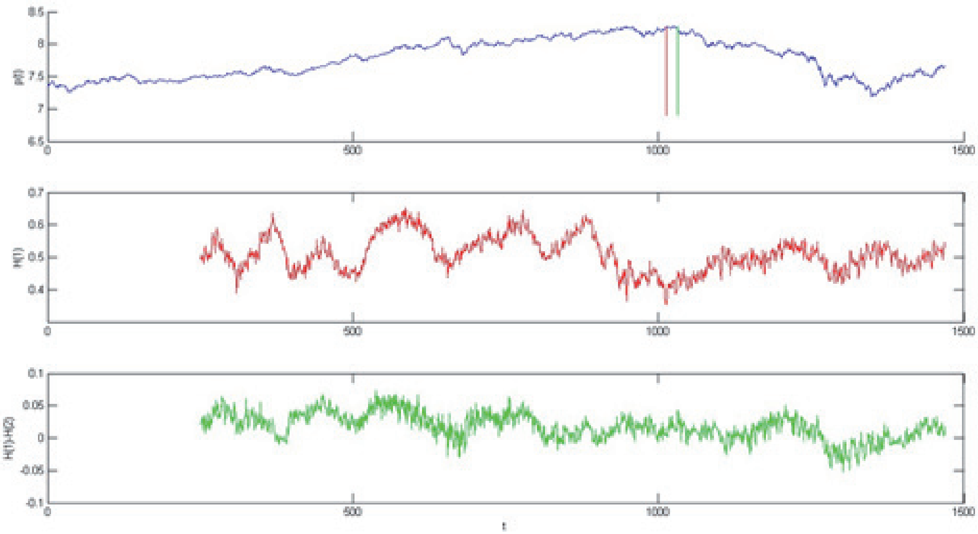
Source: Authors' own computations

Successive columns contain respectively: time series starting date t_1, its ending date t_{end}, data length n, date of local $H(1)$ minimum, local minimal value of $H(1)$, time of multifractality local maximum - max ΔH, local multifractality maximum, $H(1) - H(2)$, peak time, crash time.

For the past bubbles maximal multifractality values were obtained together with minimal GHEw prior to the peak, which stands in accordance with the aforementioned decorrelation phenomenon.

The figure below presents the specific case study with: $H(1)$ value red (local minimum marked); multifractality - green (local maximum marked):

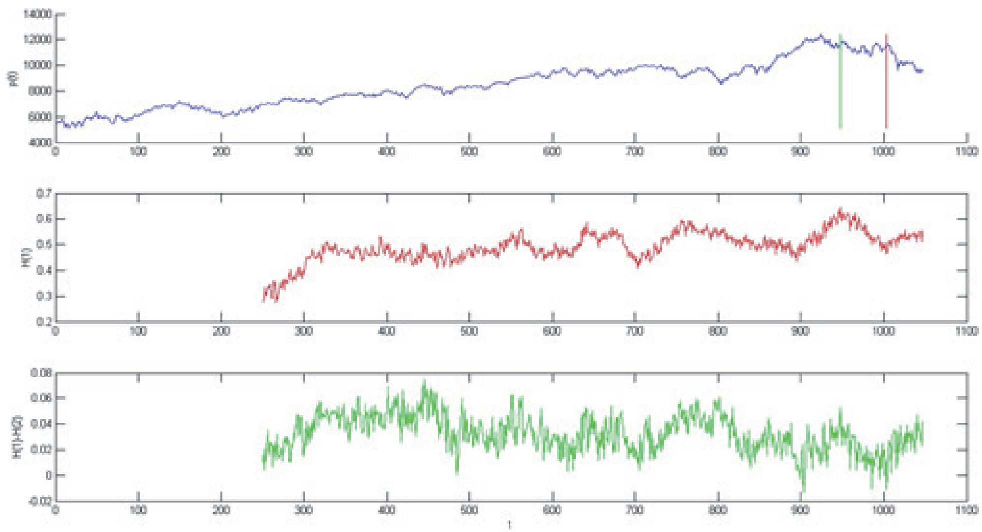
Figure 4: WIG20 3.10.2003-29.10.2007, GHEw analysis



Source: Authors' own computations

GHEw low: 10.10.2007, multifractality peak: 7.11.2007, peak: 29.10.2007, crash: 21.01.2008.

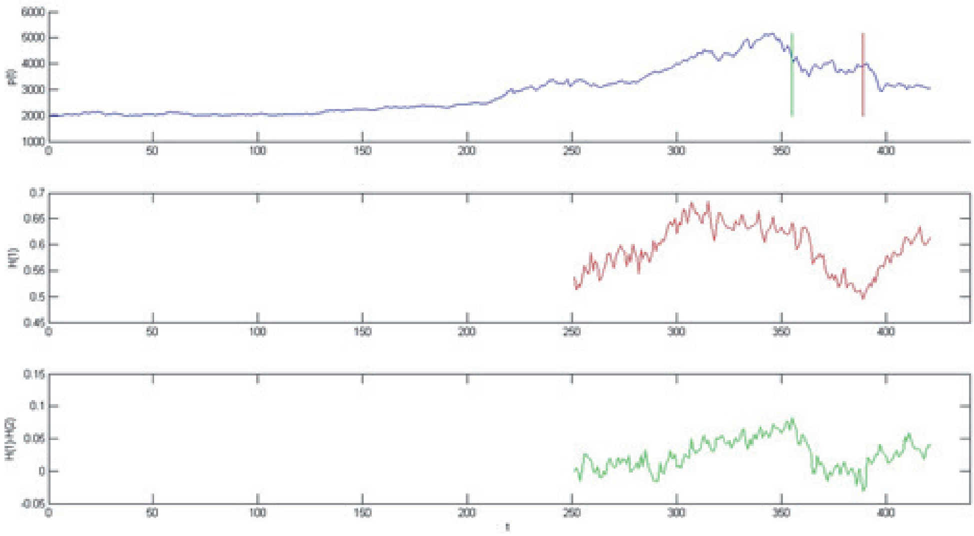
Figure 5: DAX 22.08.2011-02.10.2015, GHEw analysis



Source: Authors' own computations

GHEw low: 04.08.2015, multifractality peak: 08.05.2015, peak: 10.04.2015, crash: 24.08.2015.

Figure 6: SHX 13.01.2014-30.09.2015, GHEw analysis



Source: Authors' own computations

GHEw low: 13.08.2015, multifractality peak: 26.06.2015, peak: 12.06.2015, crash: 19.06.2015.

Our results shown above are consistent with Grech and Pamuła (2008), where the authors calculated local Hurst exponents using Detrended Fluctuation Analysis for WIG and DJIA indices, whereas in Kristoufek (2010) PSX index was considered. According to Fractal Market Hypothesis (FMH) investors have heterogeneous time horizons and strategies, which results in a fractal structure of the market. The observed multifractality evolution over time is presumably induced by changing or fitting the investment strategies. Once the specific strategy becomes predominant, the market multifractality grows, causing supply-demand disruptions and leading to the critical point of major reversion.

In most of the cases analyzed multifractality growth is accompanied by asubstantial drop of GHEw below 0.5 prior to the peak. It may be driven by systemic decorrelation within the return series (long memory tapering). As long as the bubble is growing, the market participants become more and more euphoric and nervous, amplifying thus the self-inducing feedback loop. This translates into more rapid time series oscillations right before t_c, which is well captured by log-periodic functions. After the bubble pops, GHEw again rises beyond 0.5, bringing higher correlations back into place (which incidentally makes risk hedging more difficult).

Conclusions

In our article we wanted to link the well described and established LPPL model with a new approach - GHEw analysis. We tested the two approaches on 10 historical bubbles of a large magnitude and then applied them to the current situation, e.g. on the German and Chinese stock markets. The results presented above prove convincingly that the tools used for estimating tc perform very well in most cases. The possibility of ex-ante prediction of the bubble bursting time based on the analysis of GHEw and multifractality poses an unquestionable advantage of the two presented approaches.

It can be conjectured that the specific decorrelation within time series returns is embedded into the characteristics of speculative bubbles. Strong interactions between investors give rise to self-inducing mechanisms propelling further growth of the bubble. Significant waning of these dependencies is necessary for emergence of a tipping point time tc and regime change resulting afterwards.

Further research topics cover e.g. quantitative modeling and unification of the conditions necessary for bubble bursting, most preferably in a multivariate setup. For instance, some explanatory processes might be taken into account, like NYSE margin debt readings, corporate stock buybacks scale data or dedicated investor sentiment surveys.

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