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GAMES IN A FOREIGN EXCHANGE MARKET AND SOLUTIONS

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Abstract

Exchange rate and its related risk management are too important for main participants in foreign exchange markets. There are many approaches developed in the literature for studying risk management say arbitrage detection, say finding replication portfolio. However, in the current paper, arbitrage opportunities are studied using the game theory perspective. This paper proposes different types of games played in a specified foreign exchange market in the presence of three exchange rates. Proposed games are exchange rate games in two cases of no arbitrage and existence of arbitrage, optimal stopping game, the arbitrage game, threshold strategies used in global game and Non-cooperative exchange rate game. Most of cases, the bang-bang rule of optimal control is used for finding the Nash equilibriums (NE). However, simulated and stochastic approximation (SA) solutions are also given. Most highlights of the current paper are: (I) considering two types of arbitrage opportunities, simultaneously, (II) translating arbitrage detection as game theory concepts, (III) solving the problem using techniques of optimal control theory. Finally concluding remarks are proposed.

JEL classification: G14, G15, G20
Keywords: Arbitrage, Bang-Bang rule, Non-cooperative game, Nash equilibrium, Optimal stopping, Optimal control, SA, Threshold strategies

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Research part

Introduction

Foreign exchange risk refers to the losses that an international financial transaction may incur due to currency fluctuations. Foreign exchange risk can also affect investors, who trade in international markets, and businesses engaged in the import/export of products or services to multiple countries. Exchange rate risk plays important role for multinational firms and firms that have financial functions such as hedging, speculating, investment, financing and even arbitrage in foreign exchange market. This paper is written to account for some interesting mathematical features of this type of risk (see: Moosa, 2003). To this end, consider three exchange rates $x = m_2/m_3$, $y = m_1/m_2$ and $z = m_1/m_3$. For example, is the exchange rate between currencies m_2 and m_1 measured as the price (in terms of m_2) of one unit of m_1. Here, some practical features (which leads to crucial problems) of this macroeconomic variable are studied. These features mostly are the arbitrage opportunity and the game theoretic structure of given specified foreign exchange market. This paper may be considered as combination of arbitrage in exchange market, game theory specification of problem and optimal control solution, which these topics are described shortly as below.

Game theory is a mathematical framework developed to address problems with conflicting or cooperating parties who are able to make rational decisions. The theory primarily deals with finding the optimal rational decision in various scenarios. Game theory is a relatively new discipline. Von Neumann, Morgenstern, and Nash were the main contributors to the development of game theory. The theory offers a wide number of applications in different fields, including economics, political science, finance, psychology, and biology, among others (see: Osborne & Rubinstein, 1994). Foreign exchange arbitrage is the simultaneous purchase and sale of currency in two different markets to exploit short-term pricing inefficiency. Triangular arbitrage involves the exchange of a currency in a series of three currency pairs over a short amount of time for a profit (see: Bjork, 2009; and Ma, 2008). Optimal Control theory is concerned with the problem of how to govern systems. Its purpose is to determine the best actions by which a given system can reach or maintain state, these control actions being submitted to particular conditions called "constraints" (see: Kirk, 1995).

Arbitrage detection in a foreign exchange market has been received considerable attentions in the literatures. Moosa (2003) proposed necessary and sufficient conditions for the arbitrage opportunities in several kinds of markets. Ma (2008) applied the AHP method for detecting arbitrage opportunities. Soon and Ye (2011) studied the currency arbitrages by binary integer programming method. Zhang (2012) considered an artificial neural network approach for arbitrage detection in a foreign exchange market. Habibi (2016) find the optimal arbitrage path in a given market using Markov chain and game theoretic approaches.

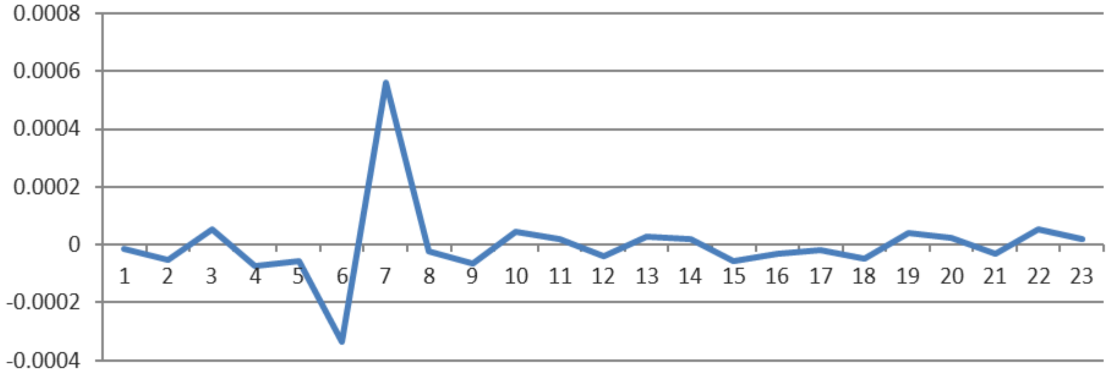
The rest of paper is organized as follows. In the next section, three types of arbitrage opportunities are proposed. Then, game theoretic aspects of exchange rate risk management problems in two cases of no arbitrage and arbitrage opportunities are proposed. Also, as a special case, threshold strategies used in global game are derived. Then, the optimal stopping game (similar Dynkin game (Kifer, 2012)), free risk game and non-cooperative games are studied. In next sections, game solutions including simulated solutions and stochastic approximation (SA) solutions of different types of exchange rate games are studied. The last section concludes.

Three types of arbitrage

The following point I, II and III studied different aspects of arbitrage opportunities in a foreign exchange market which needs a continuously monitoring, controlling and making monetary policies on different exchange rates.

(I) To avoid the arbitrage (triangular) opportunities, it is assumed that $z = xy$. Indeed, if this relation is not valid, then, following Moosa (2003), an arbitrage opportunity exists as follows. Starting one unit of currency m_1, selling m_1 and buying $1/y$ units of m_2, then selling m_2 and buying $1/xy$ units of m_3, and finally, selling m_3 and buying z/xy units of m_1 yields an arbitrage profit of $\varphi = (z/xy) - 1$. If $z = xy$ then the $\varphi = 0$. Following data used in simulation part of current paper, the time series plot of φ is as follows. It is seen that φ's are negligible.

Figure 1: Time series plot of φ



Source: Author's own work.

(II) Under the no arbitrage assumption:

$$\frac{dz}{z}=\frac{dx}{x}+\frac{dy}{y}+\frac{dxdy}{xy}$$

Consider the Ito diffusion process for x, y, z, that is:

$$\frac{dA}{A}=\mu_A dt+\sigma_A dB_A \qquad A=x,y,z$$

Suppose that cor(dx,dy) = ρ_{xy}. Then:

$$\mu_z dt+\sigma_z dB_z=\mu_x dt+\sigma_x dB_x+\mu_y dt+\sigma_y dB_y+\rho_{xy}\sigma_x\sigma_y dt$$

thus,

$$\mu_z=\mu_x+\mu_y+\rho_{xy}\sigma_x\sigma_y$$

and

$$\sigma_z dB_z=\sigma_x dB_x+\sigma_y dB_y$$

The last equation results that:

$$\sigma_z^2=\sigma_x^2+\sigma_y^2+\rho_{xy}\sigma_x\sigma_y$$

For an interesting result, consider a self-financed portfolio containing long position in x and y and short position on z, each in amount of α_A, A = x, y, z. That is:

$$P=\alpha_x x+\alpha_y y-\alpha_z z$$

Suppose that:

$$\alpha_x x\sigma_x dB_x+\alpha_y y\sigma_y dB_y-\alpha_z z\sigma_z dB_z=0$$

At the same time (α_x/y) = (α_y/x). That is, xα_x = yα_y = zα_z = P. It is concluded that there is a risk neutral probability measure such that x, y, z, P behaves like a risk free rated asset, that is dA = αAdt, A = x, y, z, P. This is the famous fundamental theorem of financial mathematics (see: Bjork, 2009).

(III) Suppose that there is a triangular arbitrage opportunity in the foreign exchange market, then z = xyƐ, for some Ɛ > 0, independent of x, y, z. Assume that:

$$\frac{d\varepsilon}{\varepsilon}=\mu_\varepsilon dt+\sigma_\varepsilon dB_\varepsilon$$

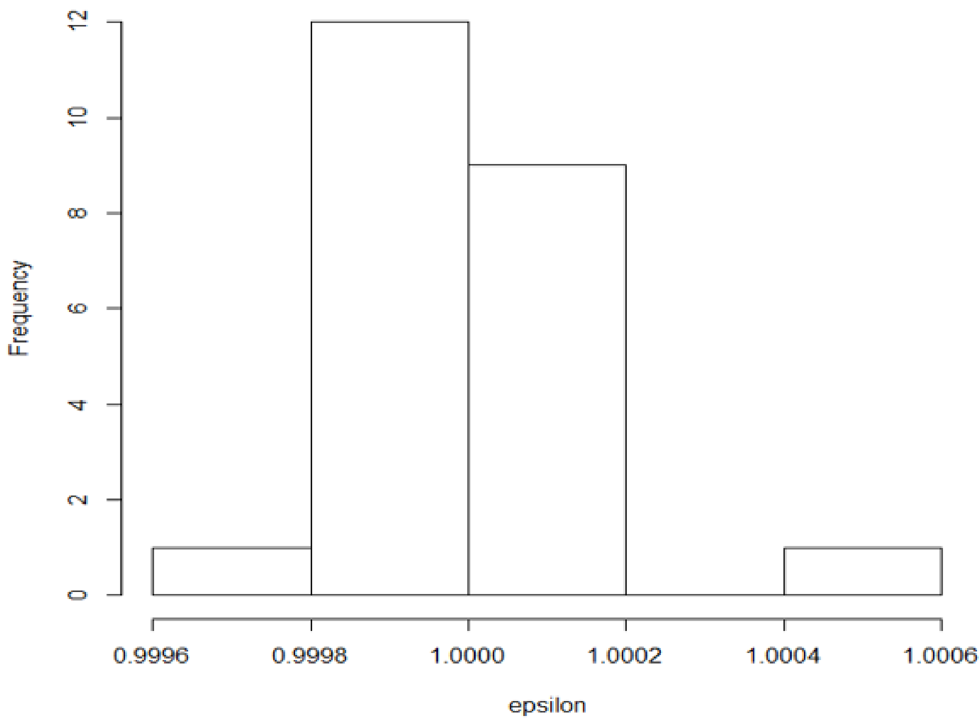
Than, φ = Ɛ - 1. Thus, log(1 + φ) = log(Ɛ). In practice, is too small and is removed by supply and demand mechanism of market, thus, it is reasonable to assume that log(1 + φ) ≈ φ.

Hence:

$$d\varphi=d\log(\varepsilon)=\frac{d\varepsilon}{\varepsilon}-\frac{\sigma_\varepsilon^2}{2}dt=(\mu_\varepsilon-\frac{\sigma_\varepsilon^2}{2})dt+\sigma_\varepsilon dB_\varepsilon$$

Practically, μ_ε and σ_ε are estimated using fitting a geometric Brownian motion to Ɛ = (z/xy). Following data used in simulation part of current paper, the following plot is histogram of Ɛ.

Figure 2: Histogram of Ɛ



Source: Author's own work.

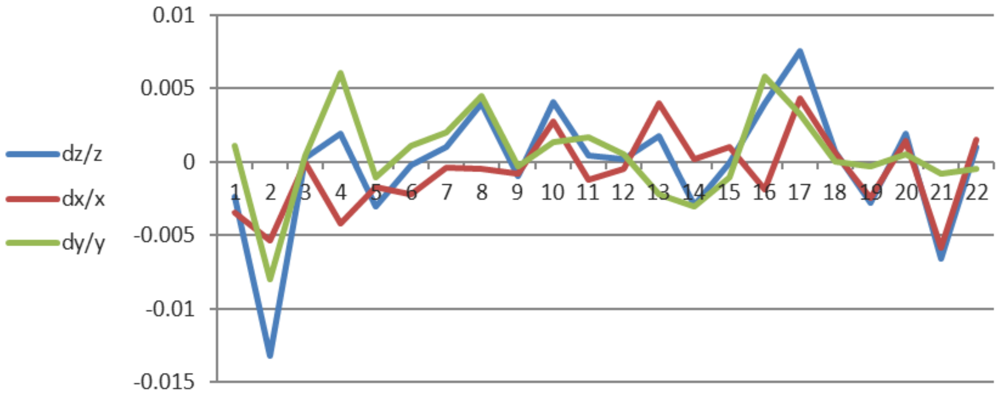
(IV) It is seen that $\mu_\varepsilon = 1$ and $\sigma_\varepsilon = 0.000146$. $E(\varepsilon^{-1}) = 0.9999$. No arbitrage assumption is satisfied. The following points studied the game theoretic structure of foreign exchange rate market.

(V) Under the no arbitrage assumption, supposing obey the Ito diffusion processes (Bjork, 2009), then one can see that:

$$\frac{dz}{z} = \frac{dx}{x} + \frac{dy}{y} + \frac{dxdy}{xy}$$

Following data used in simulation part of current paper, the following plot shows the (di/i), i = x, y, z.

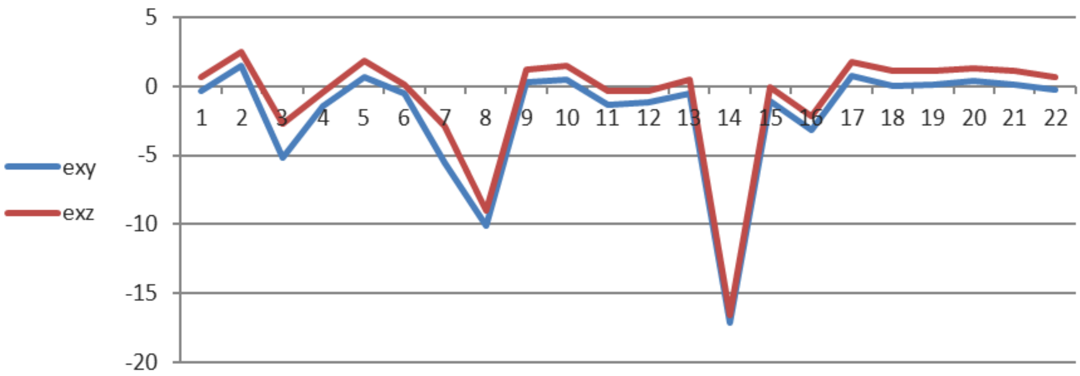
Figure 3: Time series of di/i, i = x, y, z



Source: Author's own work.

Denote the elasticity of z with respect to x by e, then $e_{xz} = 1 + e_{xy} + dy/y$. That is, fluctuations of e_{yx} will have influence on fluctuations of e_{zx}. For example, notice that, $E(e_{xz}) = 1 + E(e_{xy}) + E(dy/y)$ and assuming independence of e_{xy} and dy/y, then $var(e_{xz}) = var(e_{xy}) + var(dy/y)$. Following data used in simulation part of current paper, the following plot shows the time series plot of e_{xy}, e_{xz}.

Figure 4: Time series plot of e_{xy}, e_{xz}



Source: Author's own work.

(VI) The above point is a monetary policy. Indeed, if m_3 be desired unit money (for example, for our country) to avoid currency depreciation it is not enough to appreciate only with respect to m_2, but it is necessary to appreciate it with respect to m_1. As well as, exchange rate between m_1 and m_2 play important role. Finally, the main important is that if the central bank of desired country has pegged itself to m_2, then speculation attacks to m_3 will causes similar speculation attacks to m_1. Thus, the currency portfolio of mentioned central bank will be attacked and destroyed.

(VII) Under the no arbitrage assumption and when there is only, for example, partial information F_t^y, then exchange rate z = xy is considered as a derivative on x and it is interested to study its price, i.e. $E(z|F_t^y)$. As well as, the early exercise of (trading as soon as possible) this derivative based on optimal stopping technique is interesting. However, there are two optimal stopping criteria, i.e. $max_{T_y}E(Z_{T_y}|F_t^y)$ and $max_{T_x}E(z_{T_x}|F_t^x)$, which, again, it defines a game theoretic aspect in a foreign exchange market. This problem can be proposed for any derivative defined based z. Also, it is interested to know if there is partial information F_t^x and F_t^y, again, the no arbitrage assumption holds, if originally (when there is no partial information) hold? These problems also motivate the game theoretic aspects of exchange rate market.

Methodological part

Exchange game

In this section, different types of exchange rate games are proposed and its SA solution is given. At first, suppose that the desired currency is m_3. Then, suppose that two types of price makers exist. The first who buy and sell m_1 and others trade m_2. Thus, they produce information about exchange rates x, y. Suppose that F_t^x, F_t^y are information made by these types of price makers where F_t^x, F_t^y the σ-fields are made by $\{\chi_s, s \le t\}$, $\{y_s, s \le t\}$, respecively.

(a) No arbitrage case. Indeed, under the no arbitrage assumption z = xy, the exchange rate is affected by two exchange rates x, y. Here, the value of z is interesting. When, the information is complete, then the price of z is z, since $E(z|F_t^z) = z$. Here, F_t^z the σ-fields are made by $\{z_s, s \le t\}$. However, following Kyle (1985), suppose that there are two partial-informed traders and since there is no triangular arbitrage, thus, there is no noise trader. Again, following Kyle (1989), there price of the value (price) of z at time t, based on information of first partial-informed traders is $p_1^t = E(z|F_t^z) = xE(y|F_t^x) = xq_1^t$ where $q_1^t = E(y|F_t^x)$. Notice that assuming and using the Markov property of x, then $p_1^t = E(z|\chi)$. Analogously, p_2^t and q_2^t are defined. The profit (utility) of trader type 1 is $z - p_1^t = \chi(y - q_1^t)$. Following Aase et al. (2010), let α_t, β_t be trading intensities (volume of trades) of first and second traders, respectively. Then, the instant profit of first trader is $\pi_1 = \pi_{1t} = \alpha\chi(y - q_1^t)$ and the cumulative profit up to time t is given by $\Pi_1 = \int_0^t \pi_{1s} ds$. Both traders try to maximize π_i(or Π_i), i = 1,2, simultaneously with respect to α, β. This defines a game theoretic perspective of a given specified foreign exchange market.

The formal mathematically definition of this game to maximize cumulative profit $max_\alpha\Pi_1(x, y)$ and $max_\beta\Pi_2(x, y)$, with respect to α, β simultaneously, at which, for some trading maturity T at future, is given by the following system of equations:

$$\begin{Bmatrix} max_\alpha \Pi_1(x,y) = max_\alpha \int_0^T \alpha_t (z - E(z|F_t^x))dt, \\ max_\beta \Pi_2(x,y) = max_\beta \int_0^T \beta_t (z - E(z|F_t^y))dt \end{Bmatrix}$$

Equivalently, to maximize:

$$max_\alpha \Pi_1(x,y) = max_\alpha \int_0^T \alpha x(y - q_1^t)dt,$$

$$max_\beta \Pi_2(x,y) = max_\beta \int_0^T \beta y(x - q_2^t)dt$$

Simultaneously. It will be seen that, when are not independent, the non-zero NE's are defined by a region:

$$\{(x^*, y^*)|x > E(x|y^*), y > E(y|x^*)\}$$

Remark 1. Notice that, when the target is the economic stability (for example, when government is substituted with traders), functional max is replaced with min. However, the solution strategies do not differ.

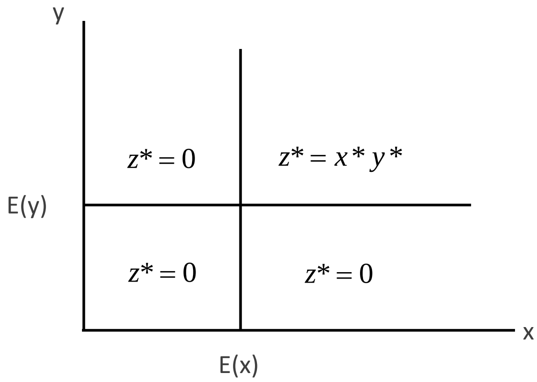
Remark 2. Suppose that x, y are statistically independent. Then, $\pi_1 = \alpha x(y - E(y))$ and $\pi_2 = \beta y(x - E(x))$, where E(x) and E(y) are expectation of x, y respectively.

Using the bang-bang optimal control (see: Kirk, 1995), to maximize instantly profit π_i, i = 1.2 it is enough to assume that:

$$\alpha = \begin{Bmatrix} 1 & y > E(y) \\ 0 & o.w \end{Bmatrix} \quad and \quad \beta = \begin{Bmatrix} 1 & x > E(x) \\ 0 & o.w \end{Bmatrix}$$

The NE occurs at observed exchange rates x*, y* if they are bigger than their mean and then α = β = 1. In this case, z* = x*y*. Otherwise, one trader (or both) does (do) not trade and hence z* = 0. The following figure shows the NE region.

Figure 5: z* NE's for various values of observed x*, y*



Source: Author's own work.

The probability of occurring non-zero NE is given by:

$$P([x > E(x)] \cup [y > E(y)])$$

Generally, motivated by remark 2, the bang-bang optimal control is given by:

$$\alpha = \begin{Bmatrix} 1 & y > E(y|x) \\ 0 & o.w \end{Bmatrix} and \ \beta = \begin{Bmatrix} 1 & x > 3(x|y) \\ 0 & o.w \end{Bmatrix}$$

thus, the non-zero NE occurs at area

$$A = (x^*, y^*)|x > E(x|y^*), \ \ y > E(y|x^*)$$

with probabilities P(A).

Remark 3. Another formulation for exchange rate game is to use the expected instant profit. Notice that when α = β = 1, then $\pi_1 = z - p_1 = x(y - q_1^t)$. Following Kyle (1985):

$$E(\pi_1|y) = yE(x|y) - E(xq_1^t|y), \ and$$
$$E(\pi_2|x) = xE(y|x) - E(yq_2^t|x)$$

Then, the NE is derived by maximizing $E(\pi_1|y)$ and $E(\pi_2|x)$, simultaneously.

(b) Free risk case. When the no arbitrage assumption is violated, then z = xyƐ, as point v of Introduction. Exchange rate is made by a noise trader (see: Kyle, 1985). Then, again following Kyle (1985), we have $p_1^t = x\varepsilon E(y|x_t\varepsilon_t)$. Here, it is assumed that ε_t is independent of x, y, thus $E(y|x_t\varepsilon_t) = q_1^t$. The first partial-informed trader uses:

$$z - x\varepsilon q_1^t = x\varepsilon(y - q_1^t)$$

to obtain gain. Having information about value of Ɛ, this game is solved similar to game of no arbitrage case. However, this assumption is not ture, in practice. Following Kyle (1985), he/she is interested to maximize expected profit:

$$\pi_1 = E(\alpha x\varepsilon(y - q_1^t)|y_t) = \alpha(y - q_1^t)E(x|y)E(\varepsilon)$$

The expected profit of second trader is:

$$\pi_2 = \beta(x - q_2^t)E(y|x)E(\varepsilon)$$

then, α = 1 if $y - q_1^t > 0$ and zero, otherwise. Also, β = 1 if $x - q_2^t > 0$ and zero, otherwise.

Here, to solve the game, it is necessary to have additional information about dynamic process affecting on the game structure. In this section, following part (V) of Introduction, it is assumed that:

$$\frac{d\varepsilon}{\varepsilon} = \mu_3 dt + \sigma_3 dB_\varepsilon$$

I Non-cooperative games. Previous games can be interpreted as cooperative game which leads to speculative attacks (see: Morris & Shin, 2006). However, in a non-cooperative game format, each trader blabs behavior of other trader. That is, the first trader opinion about z when the second trader makes information:

$$F_t^y \text{ is } E(z|F_t^y) = yE(x|F_t^y) = yE(x|y)$$

Then, his/her profit is $\pi_1 = \alpha y(x - E(y|x))$ and the profit of second player is $\pi_2 = \beta x(y - E(y|x))$. The best response of first player when the second player plays his/her best action y* is:

$$\alpha = \begin{Bmatrix} 1 & x > E(x|y^*) \\ 0 & o.w \end{Bmatrix}$$

Similarly, the best response of player 2 is:

$$\beta = \begin{Bmatrix} 1 & y > E(y|x^*) \\ 0 & o.w \end{Bmatrix}$$

Generally, the non-zero NE's are defined by a region:

$$\{(x^*, y^*) | x > E(x|y^*), y > E(y|x^*)\}$$

Remark 4. Suppose that x, y are statistically independent. Then, $\alpha = \beta = 1$ and the NE of exchange rate z, i.e., $z^* = x^*y^*$ if $x^* > E(x)$ and $y^* > E(y)$, otherwise $z^* = 0$.

(d) Optimal stopping games. Here, under the no arbitrage assumption, based on optimal stopping technique (see: Shiryaev, 2008), game structure of $\max_{T_y} E(z_{T_y}|F_t^y)$ and $\max_{T_x} E(z_{T_x}|F_t^x)$ are studied. This game is very similar to Dynkin game (option game). Following Shiryaev and Zhitlukhin (2013), the dynamic programming based backward induction in a discrete time setting implies that:

$$V_t(x) = \max(xy_t, E(V_{t-1}(xy_t)|F_t^y)), t = 1, \ldots, T$$

such that $V_0(x) = xy_0$ and:

$$W_t(y) = \max(x_t y, E(W_{t-1}(x_t y)|F_t^x)), t = 1, \ldots, T$$

such that $W_0(y) = x_0 y$. The best response functions are:

$$\begin{Bmatrix} \tau_x = \inf\{t, V_t(x) - xy^*\} \\ \tau_y = \inf\{t, W_t(y) = x^* y\} \end{Bmatrix}$$

(e) Threshold strategies. Again, under the no arbitrage assumption, and motivated by threshold strategies used in global game of Morris and Shin (2006), instant profits may be substituted by $\pi_1 = \alpha x(y - q_1^t + k)$ and $\pi_2 = \beta y(x - q_2^t + k_2)$. The intuition is that traders want more profits. The probabilities of non-zero exchange rates for the first and second traders are $P(y > q_1^t + k_1)$, $P(x > q_2^t + k_2)$, respectively. Expected non-zero based on point of views of two players are $xyP(y > q_1^t + k_1)$, and $xyP(x > q_2^t + k_2)$, respectively. Thus, are chosen such that $P(y > q_1^t + k_1)$ and $P(x > q_2^t + k_2)$ are maximized. These values are NE's selections of thresholds.

(f) Arbitrage game. In this section, the game theoretic aspects of arbitrage phenomenon in a foreign exchange market are studied. To this end, let $u = xy$ and F_t^u denotes the σ-field by $\{u_s, s \le t\}$. When, there is no arbitrage, then $u = z$, however, when there is an arbitrage opportunity, then $E(z|F_t^u) \neq z$ and difference $z - E(z|F_t^u) \neq 0$ makes an arbitrage opportunity. To describe the arbitrage game, suppose that all traders have Japanese Yen and finally, interested to attain Euro, at the trading maturity. Suppose that x = (USD/JPY), y = (EUR/USD), z = (EUR/JPY) are exchange rates. However, there are two types of buyers. Some of them use z to buy Euro, directly and some of them buy Euro, triangularly. That is, first buy USD and (using x) and then buy Euro. The combined exchange rate is $u = xy$. Let F_t^A denote the σ-field generated by random variable A, for time span (0, t).When, there is no arbitrage opportunities, then, $z = xy$. However, it is not true, in practice. Let γ_t, Θ_t denote the trading intensities (trading volumes) of first and second type categories of players. Then, their instant profits are:

$$\pi_1(z,u) = \gamma_t(z - E(z|F_t^u))$$

$$\pi_2(z,u) = \theta_t(u - E(u|F_t^z))$$

Naturally, the cumulative profits are:

$$\Pi_i(z,u) = \int_0^T \pi_i(z,u)dt, i = 1.2$$

This defines a game theoretic perspective to arbitrage opportunity in foreign exchange market. Thus, it is interested to maximize $\pi_1(z, u)$ and $\pi_2(z, u)$, with respect to γ_t, Θ_t, respectively. Again, $\Theta_t = 1$ if $u - E(u|F_t^z) > 0$ and zero otherwise. Also, $\gamma_t = 1$ if $z - E(z|F_t^u)$ and zero otherwise.

Remark 5. Another formulation for arbitrage opportunity is to write $z = xy\varepsilon$. Then:

$$p = E(xy|z) = E(\frac{z}{\varepsilon}|z) = zE(\varepsilon^{-1})$$

Assuming ε, z are independent and $E(\varepsilon^{-1})$ exists. Then:

$$\pi = xy(1 - E(\varepsilon^{-1})\varepsilon)$$

Following Kyle (1985), suppose that $x = x(z)$, $y = y(z)$. Also, notice that:

$$E(\pi|xy) = xy(1 - E(\varepsilon)E(\varepsilon^{-1}))$$

Then, the NE's selection of x, y are x*, y* (observed value) if $E(\varepsilon)E(\varepsilon^{-1}) < 1$. Otherwise, one of x*, y* (or both) is (are) zero.

Numerical results & conclusion

Game solution

In this section, the solutions of proposed game are given. Two types of solutions are given. The first type is simulated solutions. The second category of solutions is

SA solutions. Of course, before proposing SA solutions, first, some preliminaries in SA are given.

Simulations. In this section, throughout pre-required actions and initial simulations, NE's are simulated.

The following point shows the necessity of risk management in foreign exchange market.

(I) There are many risk measure for exchange risk management. Indeed, the probability p of keeping exchange rate x_t below the reasonable level U - M during time period $[t_0, t_0 + T]$ while it is touched the bad level U during period $[0, t_0]$ is a good criterion for success in managing the exchange rate risk. Indeed:

$$p = P(max_{t_0 \le t \le t_0+T} x_t < U - M \,|\, max_{0<t<t_0} x_t > U)$$

here:

$$p = P(max_{t_0 < t < t_0+16} x_t < B \,|\, max_{0<t<t_0} x_t > A)$$

is simulated. It is assumed that T = 16, t_0 = 02Apr19 and 20mar19.

Table 1: Values of p

Cases	1	2	3	4	5		6	7
A	1.1218	1.1226	1.1235	1.1263	1.1314	1.1375	1.1422	1.1435
B	1.1181	1.1163	1.1224	1.1252	1.1355	1.1391	1.1445	1.1488
p	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.6535	0.9934

Source: Author's own work.

The following point shows the necessity of auditing of model used in modeling the risk of foreign exchange rates.

(II) An error in computing exchange rate leads to terrible faults. To this end, let the percentage of change of x_t be $l_t = (x_t - x_{t-1})/(x_{t-1})$. Then, $x_t = (1 + l_t)x_{t-1}$. Therefore, $x_t - x_{t-1} = l_t x_{t-1}$. Therefore:

$$x_t = x_0 + \sum_{j=2}^{t} l_j x_{j-1}$$

Assume that at some time, $2 \le k \le t$, l_k is computed as l_k^*. Then, what happens for x_t? What is the effect of $d_k = l_k - l_k^*$ in computations?

Notice that:

$$x_t^* = x_0 + \sum_{j=2, j\ne k^*}^{t} l_j x_{j-1}$$

Therefore,

$$x_t - x_t^* = (l_k - l_k^*)x_{k-1} = ((1+l_k)-(1+l_k^*))x_{k-1} = x_k - x_k^*$$

hence,

$$\frac{x_t - x_{t-1} - (x_t^* - x_{t-1})}{x_{t-1}} = \frac{x_k - x_{k-1} - (x_k^* - x_{k-1})}{x_{k-1}} \times \frac{x_{k-1}}{x_{t-1}}$$

Therefore, since log(1 + x) ≈ x for small x's, then it is seen that:

$$d_t = d_k \left(\frac{x_{k-1}}{x_{t-1}}\right) = d_k \prod_{j=k}^{t-1} (1+I_j)^{-2} \approx$$

$$d_k \exp\left\{-\sum_{j=k}^{t-1} \log(1+I_j)\right\} = d_k \exp\left\{-\sum_{j=k}^{t-1} I_j\right\}$$

That is:

$$e^{\sum_{j=1}^{t} I_j} d_t$$

and constant over time passes.

Next, it is interested to find the conditional expectations. To do so, x, y first are simulated from Ito processes (as mentioned in point (II) of Introduction) with correlation coefficient ρ_{xy}. Indeed, it is enough to write

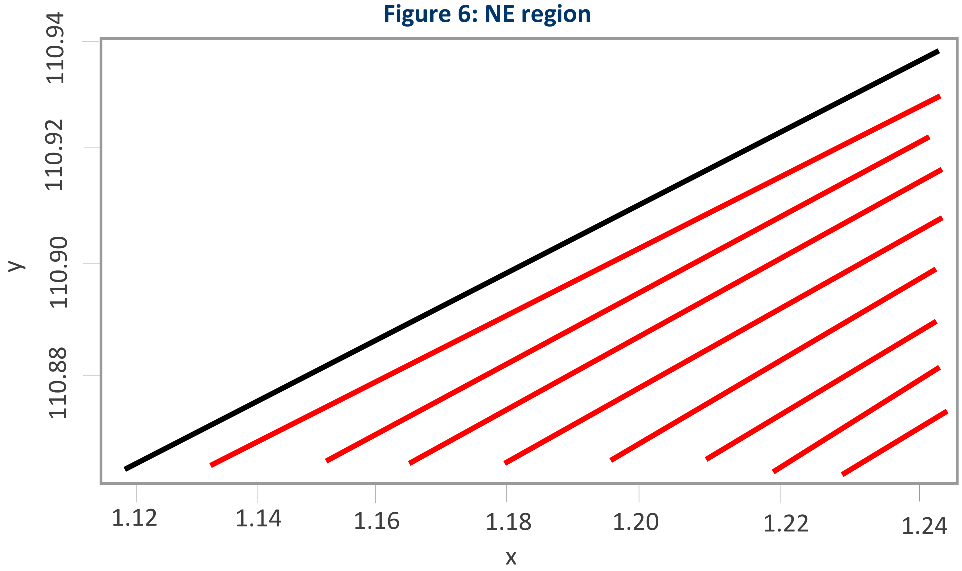
$$\frac{dA}{A} = \mu_A dt + \sigma_A dB_A,\ A = x, y$$

and assume that:

$$dB_y = \sqrt{1-\rho_{xy}^2}\, dB + p_{xy} dB_x$$

where dB and dB_x are two independent Brownian motions. Then, sample mean of x is derived for some y's such that for a specific y_δ, then $|y - y_\delta| < \delta$, for some small δ (say, 0.1). Then, a regression between various sample means and y_δ, gives the relationship of E(x|y). It is concluded that E(x|y) = -192.89 + 1.75y. Similarly, it is seen that E(x|y) = 1.1061. Then, the NE occurs at x + 192.89 > 1.75y and y > 1.1061. The following figure gives the non-zero NE's region.

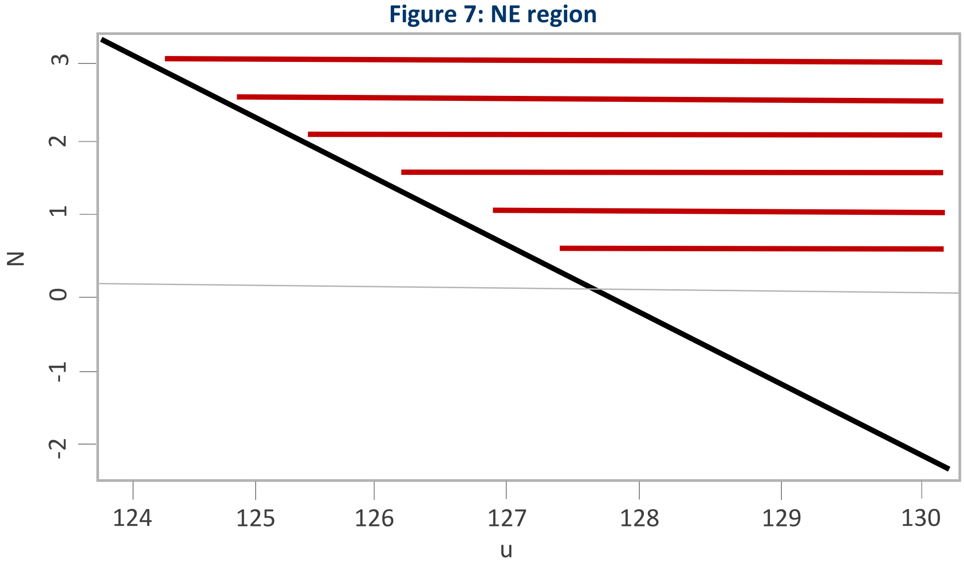
Figure 6: NE region



Source: Author's own work.

Probability of non-zero exchange rate is 0.998. Hereafter, the NE solutions of k_i, i = 1,2 of threshold strategies game are simulated. Notice that $q_2^t = -192.89 + 1.75y$. Then maximized $P(x > q_2^t + k_2)$ is achieved by $k_2 = 0.3173525$. Also, the NE selection of k_1 is given by 5.791389. Also, the arbitrage game is simulated. It is seen that E(z|u) = -7610.3 + 60.77u and E(u|z) = -1104.5061 + 9.0252z. The following figure gives the NE region.

Figure 7: NE region



Source: Author's own work.

The main question of paper was how to find the NE. Really, it is easy in two players and finite sample of actions and bi-matrix payoffs. However, it is a difficult task when numbers of players or action spaces of players are large. There are many strong theorems such as min-max and max-min theorems to find NE's in zero sum games. Indeed, finding NE's in symmetric games is too simpler than general form of games; see Osborne and Rubinstein (1994). The Lemke-Howson method gives an algorithmic solution to find NE's, see Lemke and Howson (1964).

There are many strong fixed point theorems to check the existence of NE's, see Border (1999). SA method proposes an iterative method to solve equations when there are no analytical solutions (see: Fudenberg & Tirol, 1991). The stochastic approximation (hereafter SA) method gives:

$$x_{i+1} = (1-\lambda_i)x_i + \lambda_i b_1(y_i)$$

$$y_{i+1} = (1-\gamma_i)x_i + \gamma_i b_2(x_i)$$

Generally, the above sequences converge to x*, y* if:

$$\lambda_i, \gamma_i \geq 0, \sum_{i\geq 1} \lambda_i, \gamma_i = \infty, \text{ and } \sum_{i=1}^{\infty} \lambda_i^2, \gamma_i^2 < \infty$$

A regular choice for λ_i, γ_i is 1/i, see Borkar (2008). The continuous time type of above equations is:

$$\begin{Bmatrix} x_t' = -x_t + b_1(y_t), \\ y_t' = -y_t + b_2(x_t) \end{Bmatrix}$$

Consider the Cournot duopoly game. Assume that i-th agent produces quantity $q_i \in [0, a)$, i = 1, 2. The total cost of i-th agent is cq_i, i = 1,2 where c < a. Thus, the utility of i-th agent is:

$$u_i(q_1, q_2) = (a - c - (q_1 + q_2))q_i, \; i = 1, 2$$

The continuous type of SA method with $\lambda_i = \gamma_i = 1/i$ is given:

$$q' = -(A + \frac{I}{2})q + \frac{a-c}{2}1$$

$$\text{Where, } q = (q_1, q_2)^T$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$1 = (1,1)^T$$

Vectors are bolded and I is an 2 x 2 identity matrix. Notation T stands for transpose of a vector or a matrix.

The results of previous sections may be considered as trading strategies, since trading rules α, β are determined. However, to determine the NE's of x,y, notice that:

$$\pi_1 = x(y - q_1(x)) \text{ and } \pi_2 = y(x - q_2(y))$$

From previous section, it is seen that:

$$q_1(x) = a_1 x + b_1 \text{ and } q_2(y) = a_2 y + b_2$$

Solving the best response functions obtained by equations:

$$\frac{\partial \pi_1}{\partial x} = \frac{\partial \pi_2}{\partial y} = 0$$

it is seen that:

$$x^* = \frac{1}{2a_1} y^* - \frac{b_1}{2a_1} \text{ and } y^* = \frac{1}{2a_2} x^* - \frac{b_2}{2a_2}$$

However, the linearity forms of functions q_i, i = 1,2 is true for some special forms of Ito processes. When, there are not linear functions, equations:

$$\frac{\partial \pi_1}{\partial x} = \frac{\partial \pi_2}{\partial y} = 0$$

are non-linear type and recursive numerical solutions such as Newton-Raphson or SA technique is necessary. Also, it is possible to simulate q_i, i = 1,2 using a Monte Carlo simulation. These type of recursive relations can be interpreted as learning processes of players. Using the above results, it is seen that:

$$x_{i+1} = (1-\lambda_i)x_i + \lambda_i b_1(y_i) \text{ and } y_{i+1} = (1-\gamma_i)x_i + \gamma_i b_2(x_i)$$

Where b_1 and b_2 are solutions of differential equations:

$$b_2 - q_1(b_1) - b_1 q_1'(b_1) = 0 \text{ and } b_1 - q_2(b_2) - b_2 q_2'(b_2) = 0$$

Suppose that best response functions satisfy in the differential equations:

$$b_2 - q_1(b_1) - b_1 q_1'(b_1) = 0 \text{ and } b_1 - q_2(b_2) - b_2 q_2'(b_2) = 0$$

Then, the SA solution: (a) in discrete time framework is given by:

$$\begin{Bmatrix} x_{i+1} = (1-\lambda_i)x_i + \lambda_i b_1(y_i) \\ y_{i+1} = (1-\gamma_i)x_i + \gamma_i b_2(x_i) \end{Bmatrix}$$

In continuous time setting is given by:

$$\begin{Bmatrix} x_t' = -x_t + b_1(y_t) \\ y_t' = -y_t + b_2(x_t) \end{Bmatrix}$$

Conclusion

Many types of games are defined in a given foreign exchange market in the presence of at least three exchange rates. In designing these games, it is tried to cover many interesting features of exchange rate as an important macroeconomic variable. Two main approaches to find NE solutions are simulation and SA technique. Frequently the bang-bang optimal control method is applied to find the NE's when it is interested to find the trading strategies where they are well-defined by coefficients multiplying to the profits of players. In spite of Kyle (1985), here, it is assumed that there are partial informed traders, only, in the case of no arbitrage and a noise trader, following Kyle (1985), when there are triangular arbitrage opportunities.

In many strategic situations in economics such as Cournot and Bertrand duopolies, first and second price auctions, voluntary contribution of public good, the Nash equilibrium (hereafter NE) is a solution to make predictions, see Kifer (2012). Indeed, each player plays his/her best response action versus other players. For more description, consider a two player game at which b_1 and b_2 are the best response functions of player 1 and 2, respectively. It is well known that the Nash equilibrium occurs at (x*, y*) such that $x^* = b_1(y^*)$, $y^* = b_2(x^*)$ The regular recursive relations to find x*, y* are $x_{i+1} = b_1(y_i)$ and $y_{i+1} = b_2(x_i)$ at which x*, y*, are limiting points of $\{x_i, y_i, i \geq 1\}$ if exists.

As mentioned in the previous sections, in this article, a combination of two types of multiple arbitrages was considered in the currency market over time, and first three types of arbitrage, related strategies and possible processes were described. In the game definition section, first, using the information asymmetry model in the currency market, we defined an interactive differential game that considers both multiple arbitrage and over time between the buyers and sellers of the exchange rate in the currency market. It can be seen that the Nash equilibrium of this game indicates the absence of arbitrage of both types for the participants of the currency market.

Also, this equilibrium is compatible with the two principles of the risk-free world (which is defined as the risk-free game) and the forbidden free lunch (which is defined as the arbitrage game). Then, using the general game theory (Morris & Shin, 2006), a competitive game between players was defined, and finally the Nash equilibrium was extracted in terms of threshold strategies, where the cross-sectional Nash equilibrium, if it exists, will gradually disappear with the condition of the absence of temporal Nash equilibrium. Also, using the theory of optimal stopping, Dynkin games were defined and the Nash equilibrium indicated that if there is no cross-sectional Nash equilibrium, the temporal Nash equilibrium will also be lost. For the numerical solution of the games in the simulated examples, you used S techniques and optimal control and optimal stopping. The following results are derived from this research:

a) cross-sectional and time arbitrages are interchangeable,
b) if there is no arbitrage from one variety, the other will also disappear over time,
c) it is possible that a certain game type eliminates both types of arbitrage, but the other game type does not.

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